Regular measures on tribes of fuzzy sets

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Related work presented at Linz Seminars

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1979

Henri M. Prade: Nomenclature of fuzzy measures

Erich Peter Klement: Extension of probability measures to fuzzy measures

and their characterization

Werner Schwyhla: Conditions for a fuzzy probability measure to be an integral

Josette and Jean-Louis Coulon: Fuzzy boolean algebras

1980

Erich Peter Klement: Some remarks on t-norms, fuzzy σ -algebras and fuzzy measures

Werner Schwyhla: Remarks on non-additive measures and fuzzy sets

Ulrich Höhle: Fuzzy measures as extensions

1981

Erich Peter Klement: Fuzzy measures assuming their values in the set of fuzzy numbers

Erich Peter Klement: On different approaches to fuzzy probabilities

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Didier Dubois: Upper and lower possibilistic expectations and applications

Ronald R. Yager: Probabilities from fuzzy observations

1983

Siegfried Weber: How to measure fuzzy sets

1984

Robert Lowen: Spaces of probability measures revisited

1985

Siegfried Weber: Generalizing the axioms of probability

1986

Erich Peter Klement: Representation of crisp- and fuzzy-valued measures by integrals

Siegfried Weber: Some remarks on the theory of pseudo-additive measures and its applications

Erich Peter Klement: On a class of non-additive measures and integrals

1988

Alain Chateauneuf: Decomposable measures, distorted probabities and concave capacities

Siegfried Weber: Decomposable measures for conditional objects

Aldo Ventre: A Yosida-Hewitt like theorem for \perp -decomposable measures

(joint paper with M. Squillante)

Massimo Squillante: \bot -decomposable measures and integrals: Convergence and absolute continuity (joint paper with L. D'Apuzzo and R. Sarno) Ulrich Höhle: Non-classical models of probability theory

i

1998

Mirko Navara, Pavel Pták: Types of uncertainty and the role of the Frank t-norms in classical and nonclassical logics

Mirko Navara: Nearly Frank t-norms and the characterization of T-measures Giuseppina Barbieri: A representation theorem and a Liapounoff theorem for T_s -measures

Beloslav Riečan: On the probability theory and fuzzy sets

Ulrich Höhle: Realizations for generalized probability measures

Marc Roubens: On probabilistic interactions among players in cooperative

games

Radko Mesiar: k-order pseudo-additive measures

Classical measure theory [Halmos]

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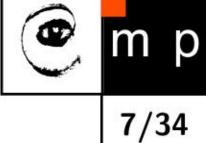
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THEOREMS about

FUNCTIONALS (MEASURES) on

SETS

Also [Sugeno; Dubois, Prade; Wang, Klir; Pap]



THEOREMS about

FUZZY FUNCTIONALS (MEASURES) on

SETS

[Feng; Guo, Zhang, Wu]

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THEOREMS about

FUNCTIONALS (MEASURES) on

FUZZY SETS

[Butnariu, Klement]

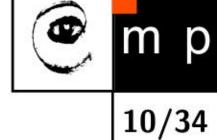
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THEOREMS about

FUZZY FUNCTIONALS (MEASURES) on

FUZZY SETS

[???]

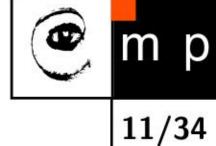


FUZZY THEOREMS about

FUZZY FUNCTIONALS (MEASURES) on

FUZZY SETS

[!!!]



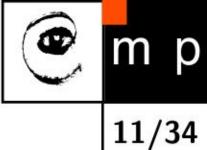
HERE:

THEOREMS about

FUNCTIONALS (MEASURES) on

FUZZY SETS

[Butnariu, Klement, Mesiar, Barbieri, Weber]



HERE:

THEOREMS about

FUNCTIONALS (MEASURES) on

FUZZY SETS

[Butnariu, Klement, Mesiar, Barbieri, Weber]

Also measure theory on MV-algebras [Cignoli, D'Ottaviano, Mundici, Riečan]



Standard negation, $\neg x = 1 - x$

12/34



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Fuzzy conjunction (t-norm): $T:[0,1]^2 \to [0,1]$ which is commutative, associative, nondecreasing, and T(a,1)=a



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A t-norm T is **strict** iff it is **continuous** and $x > y, z > 0 \Rightarrow T(x, z) > T(y, z)$

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Fuzzy disjunction (t-conorm): $S: [0,1]^2 \rightarrow [0,1]$ dual to T:

$$S(x,y) = \neg T(\neg x, \neg y)$$





classical measure theory	fuzzy measure theory
σ -algebra $\mathcal{T} \subseteq 2^X$	tribe (\mathcal{T},T) , where $\mathcal{T}\subseteq [0,1]^X$
$\emptyset \in \mathcal{T}$	$0 \in \mathcal{T}$
$A \in \mathcal{T} \Rightarrow A' = X \setminus A \in \mathcal{T}$	$A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T}$
$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$	$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}^*$
$(A_n)_{n\in\mathbb{N}}\subseteq\mathcal{T}, A_n\nearrow A\Rightarrow A\in\mathcal{T}$	$(A_n)_{n\in\mathbb{N}}\subseteq\mathcal{T}, A_n\nearrow A\Rightarrow A\in\mathcal{T}$
measure μ : $\mathcal{T} \to [0, \infty[$	measure μ : $\mathcal{T} \to [0, \infty[$
$\mu(\emptyset) = 0$	$\mu(0) = 0$
$\mu(A \cup B)$	$\mu(A \overset{\cdot}{\cup} B)$
$= \mu(A) + \mu(B) - \mu(A \cap B)$	$= \mu(A) + \mu(B) - \mu(A \cap B) *$
$A_n \nearrow A \Rightarrow \mu(A_n) \to \mu(A)$	$A_n \nearrow A \Rightarrow \mu(A_n) \to \mu(A)$

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$$(A \cap B)(x) = T(A(x), B(x)), \qquad (A \cup B)(x) = S(A(x), B(x))$$

Basic notions of fuzzy measure theory



classical measure theory $\sigma\text{-algebra }\mathcal{T}\subseteq \mathbf{2}^{X} \succeq \mathcal{V}(X) \iff \mathsf{tribe}\ (\mathcal{T},\mathcal{T}), \text{ where }\mathcal{T}\subseteq [0,1]^{X}$ $\emptyset\in\mathcal{T} \qquad 0\in\mathcal{T} \qquad 1\in\mathcal{T}$ $A\in\mathcal{T}\Rightarrow A'=X\setminus A\in\mathcal{T} \qquad A\in\mathcal{T}\Rightarrow A'=1-A\in\mathcal{T}$ $A,B\in\mathcal{T}\Rightarrow A\cap B\in\mathcal{T} \qquad A,B\in\mathcal{T}\Rightarrow A\cap B\in\mathcal{T} \qquad A,B\in\mathcal{T}\Rightarrow A\cap B\in\mathcal{T} \iff A,B\in\mathcal{T}\Rightarrow A\cap B\in\mathcal{T}$

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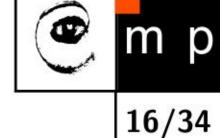




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Always: Crisp elements of \mathcal{T} , i.e., $\mathcal{T} \cap \{0,1\}^X$, determine a σ -algebra \mathcal{B}

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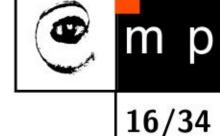


Example: Let \mathcal{B} be a σ -algebra of subsets of X,

 \mathcal{T} be the corresponding collection of characteristic functions (indicators):

$$\mathcal{T} = \{ \chi_A \mid A \in \mathcal{B} \} .$$

Then (T,T) is a tribe (for any t-norm T). It is called a **Boolean tribe**.



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Łukasiewicz t-norm



$$T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$$

These tribes correspond to set-representable σ -complete MV-algebras

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