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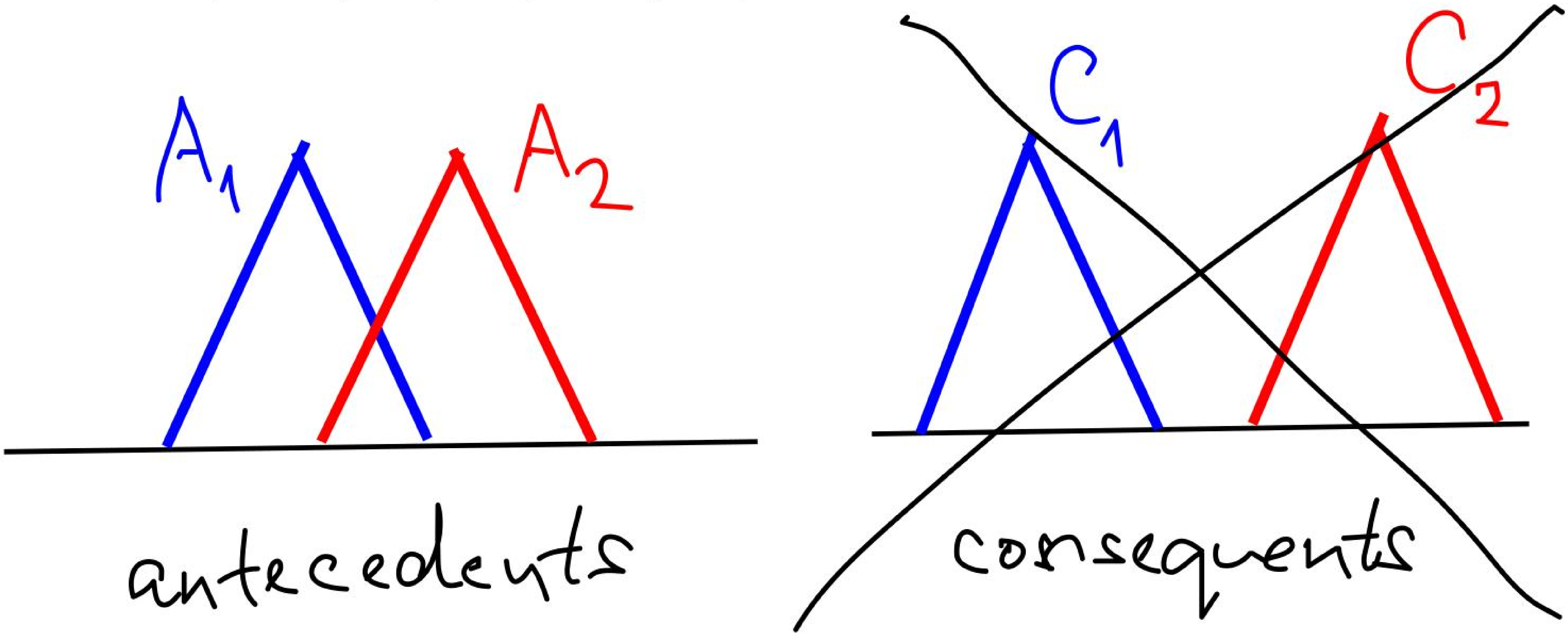

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## Recommendations on the rule base [Driankov et al. 1993]



Antecedents (one-dimensional) should be

- **normal**, $\forall i \ \exists x \in \mathcal{X} : A_i(x) = 1$,
- continuous,
- symmetric (when possible, usually not at the borders of the input space!).

The recommended degree of overlapping of neighbouring antecedents (computed using the standard t-norm, $\min$) is $0.5$.

The recommended endpoints of the support of an antecedent are the peeks of the neighbouring antecedents.

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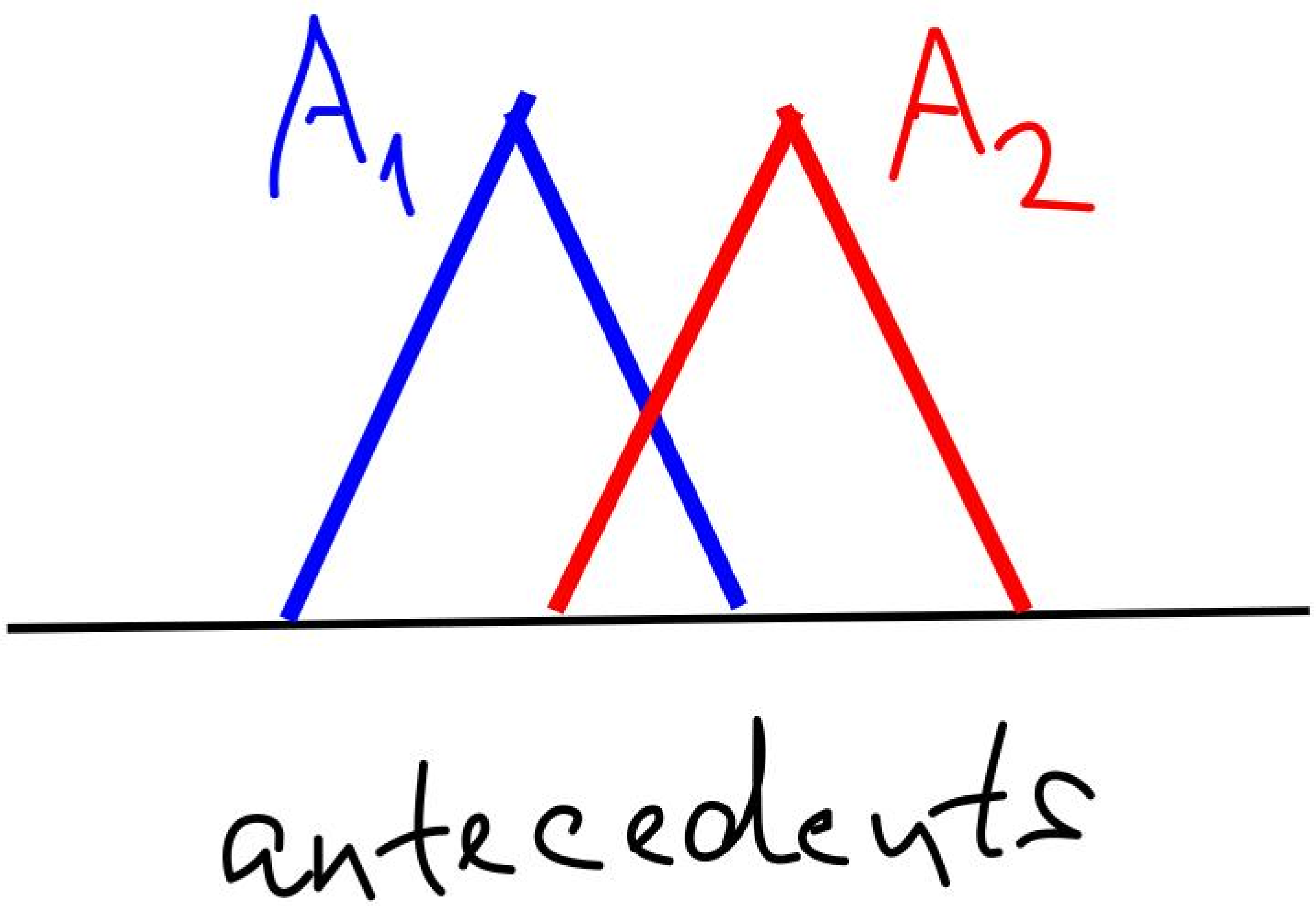


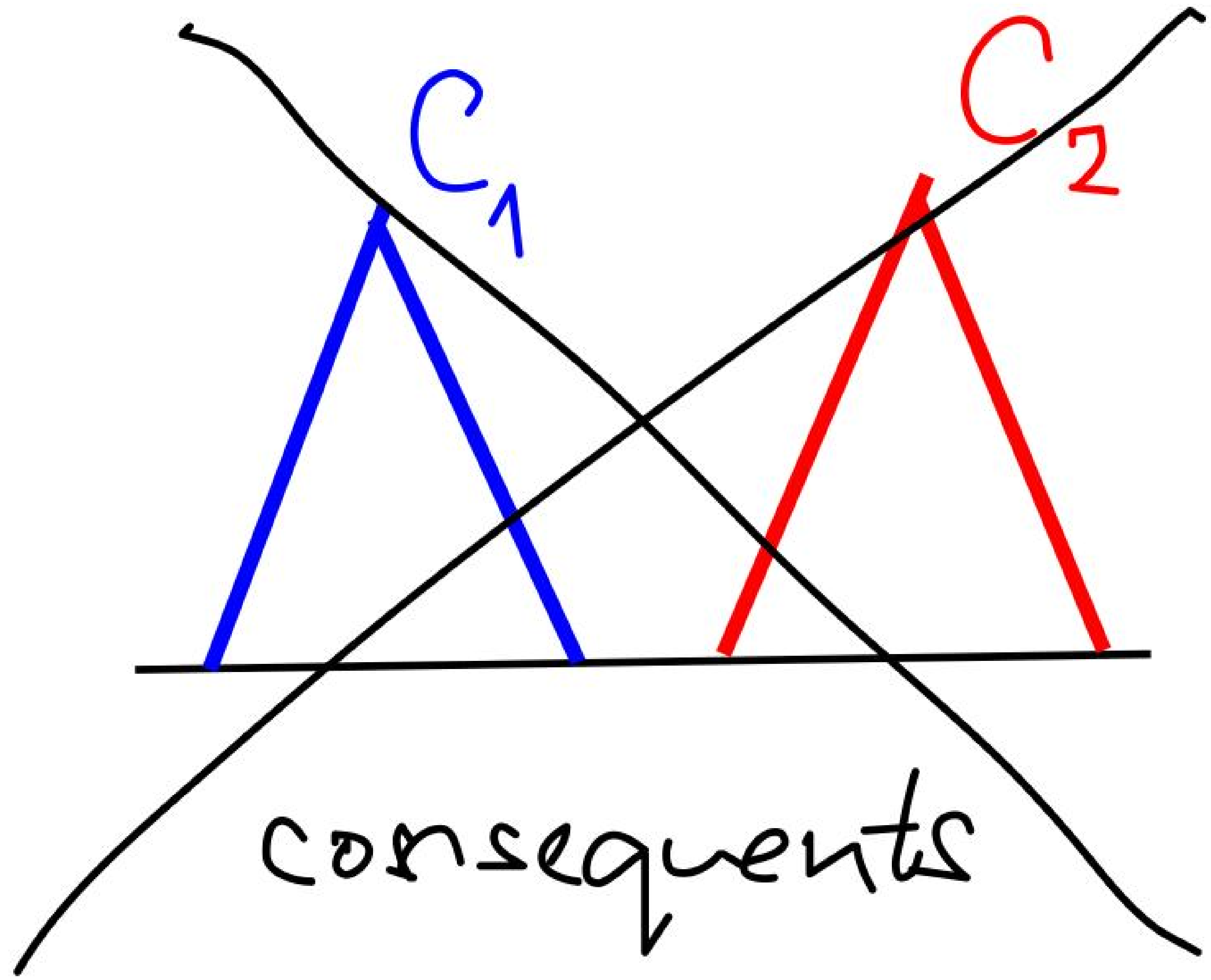

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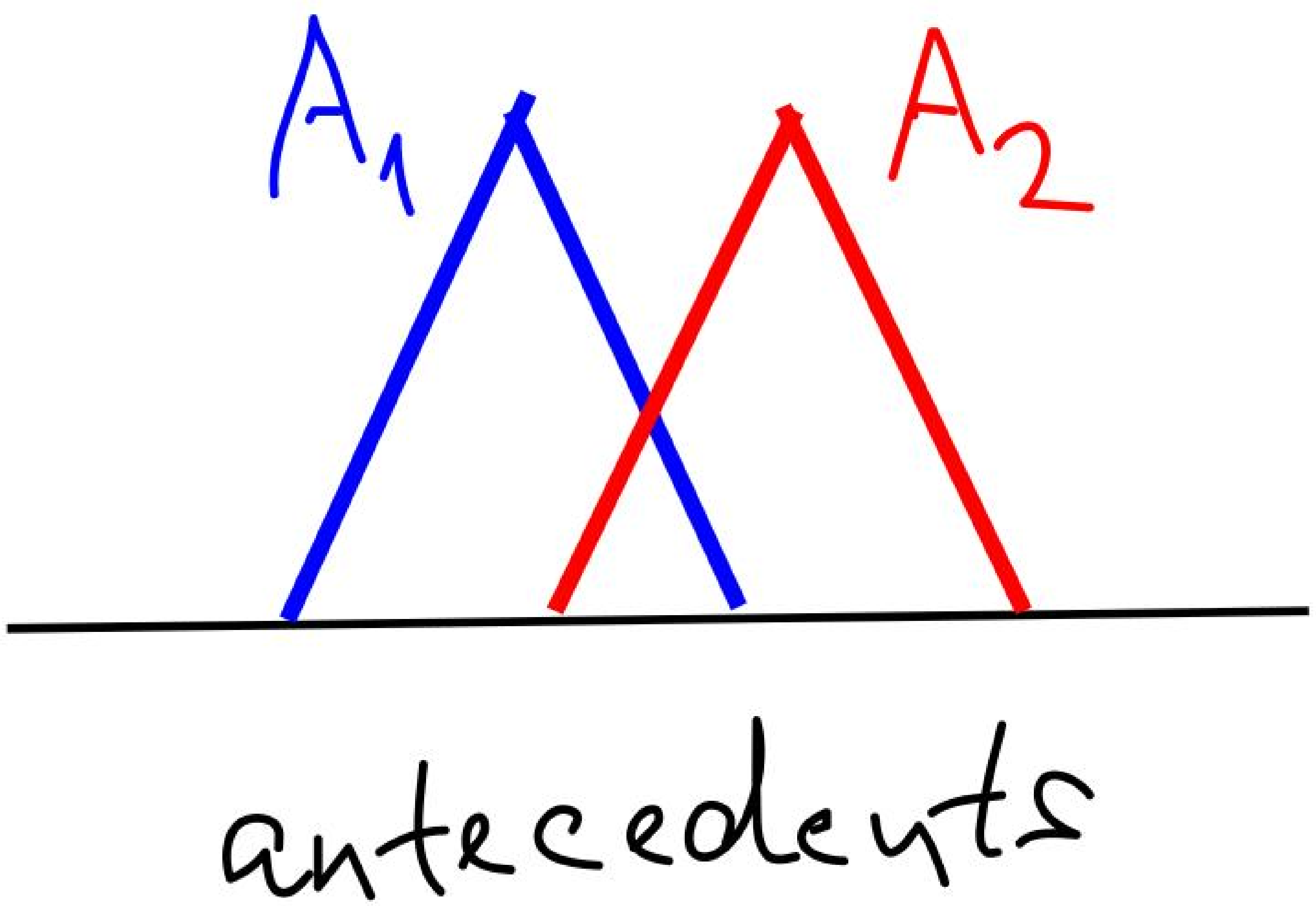


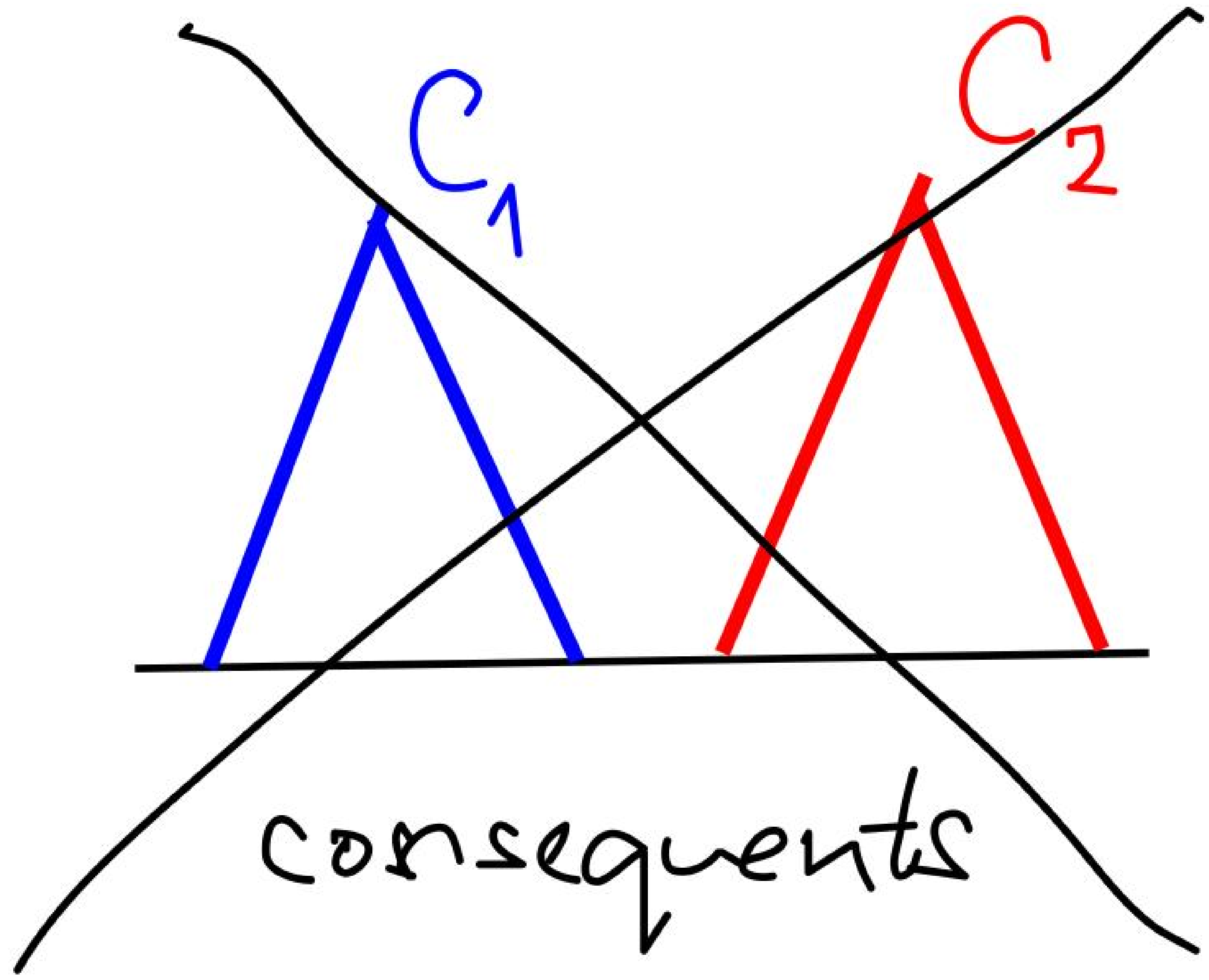

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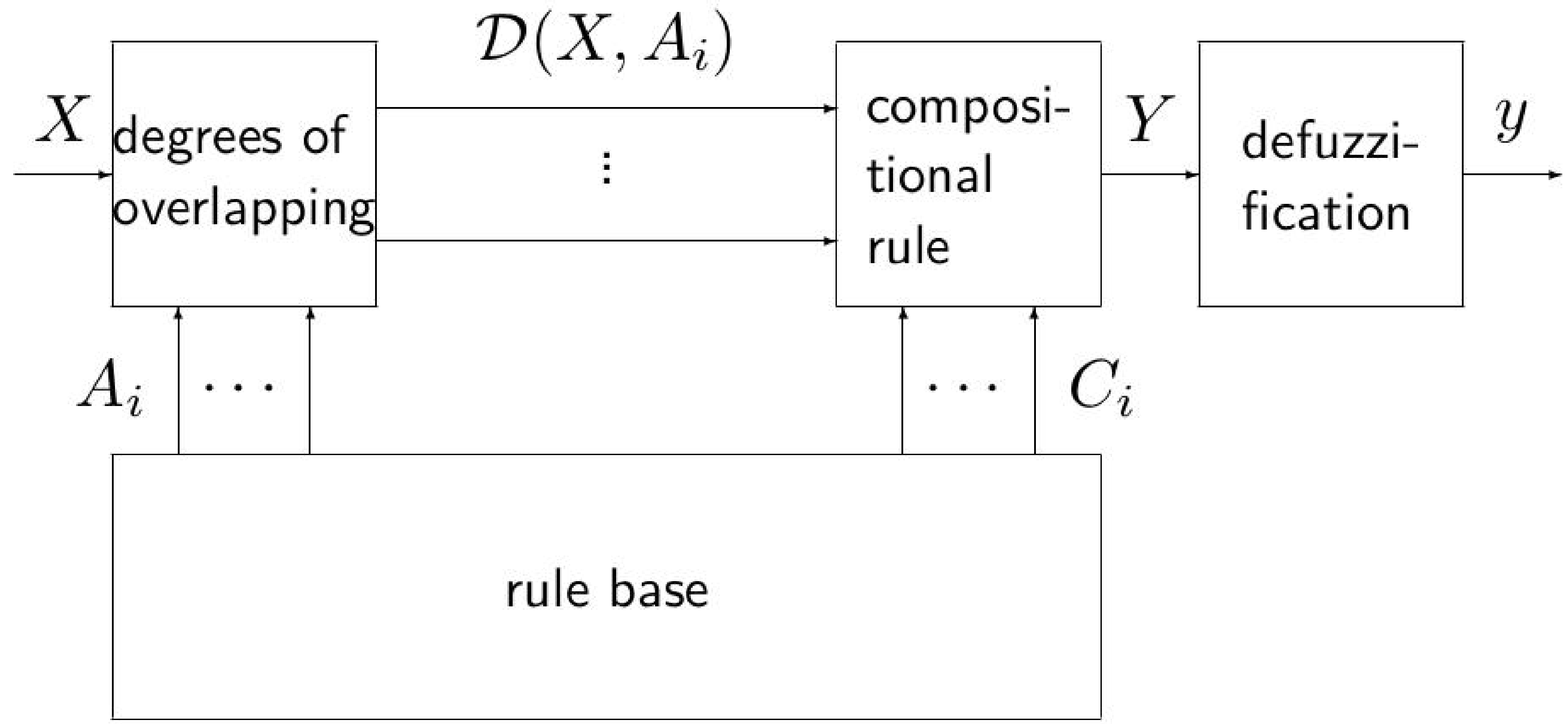

# Comparison of residuum-based and Mamdani–Assilian controllers



**Continuity:**

$R_{\mathsf{RES}}$ only for $\dot{\wedge}$ nilpotent,

$R_{\mathsf{MA}}$ always.

**Computational efficiency:**

$$\Phi_{\mathsf{RES}}(X)(y) = \sup_x \Big( X(x) \mathbin{\dot{\wedge}} \min_i (A_i(x) \mathbin{\dot{\to}} C_i(y)) \Big)$$

requires **three** nested cycles (over $\mathcal{X}$ and $\mathcal{Y}$ and over the number of rules).

$$\begin{aligned} \Phi_{\mathsf{MA}}(X)(y) &= \sup_x \Big( X(x) \mathbin{\dot{\wedge}} \max_i (A_i(x) \mathbin{\dot{\wedge}} C_i(y)) \Big) \\ &= \max_i \sup_x \Big( X(x) \mathbin{\dot{\wedge}} A_i(x) \mathbin{\dot{\wedge}} C_i(y) \Big) \\ &= \max_i (\mathcal{D}(X, A_i) \mathbin{\dot{\wedge}} C_i(y)) \,. \end{aligned}$$

$\mathcal{D}(X, A_i) = \sup_x (X(x) \mathbin{\dot{\wedge}} A_i(x))$ ... the **degree of overlapping (non-disjointness)**, here equal to the **degree of firing (applicability)**.

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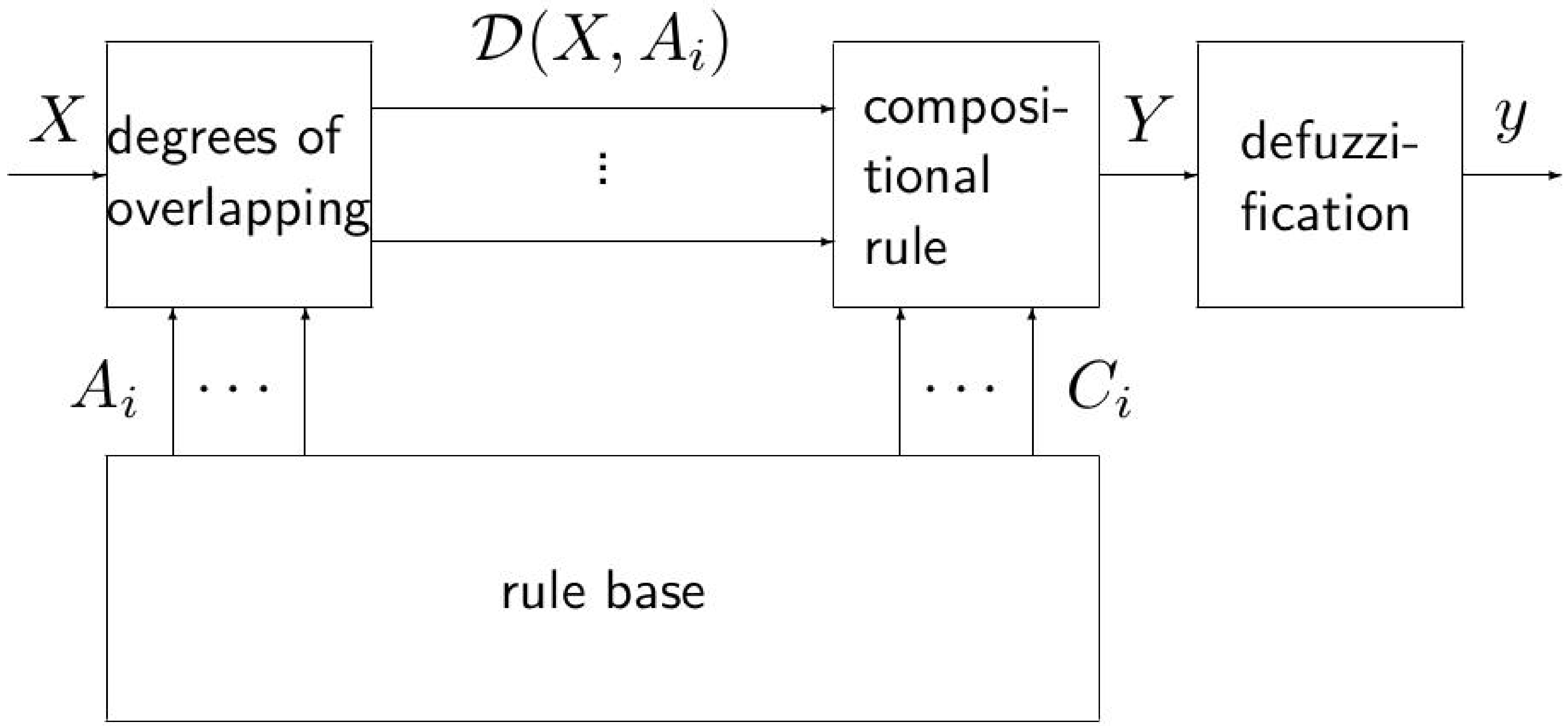

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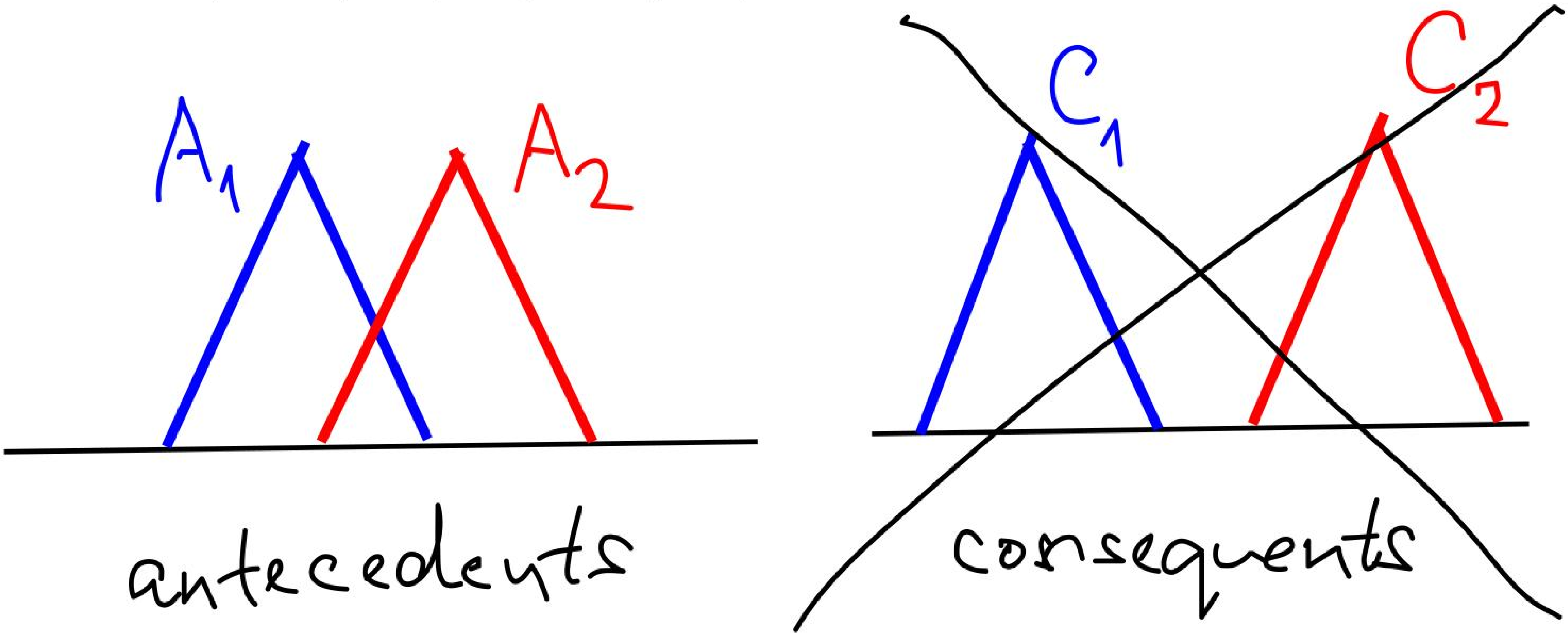

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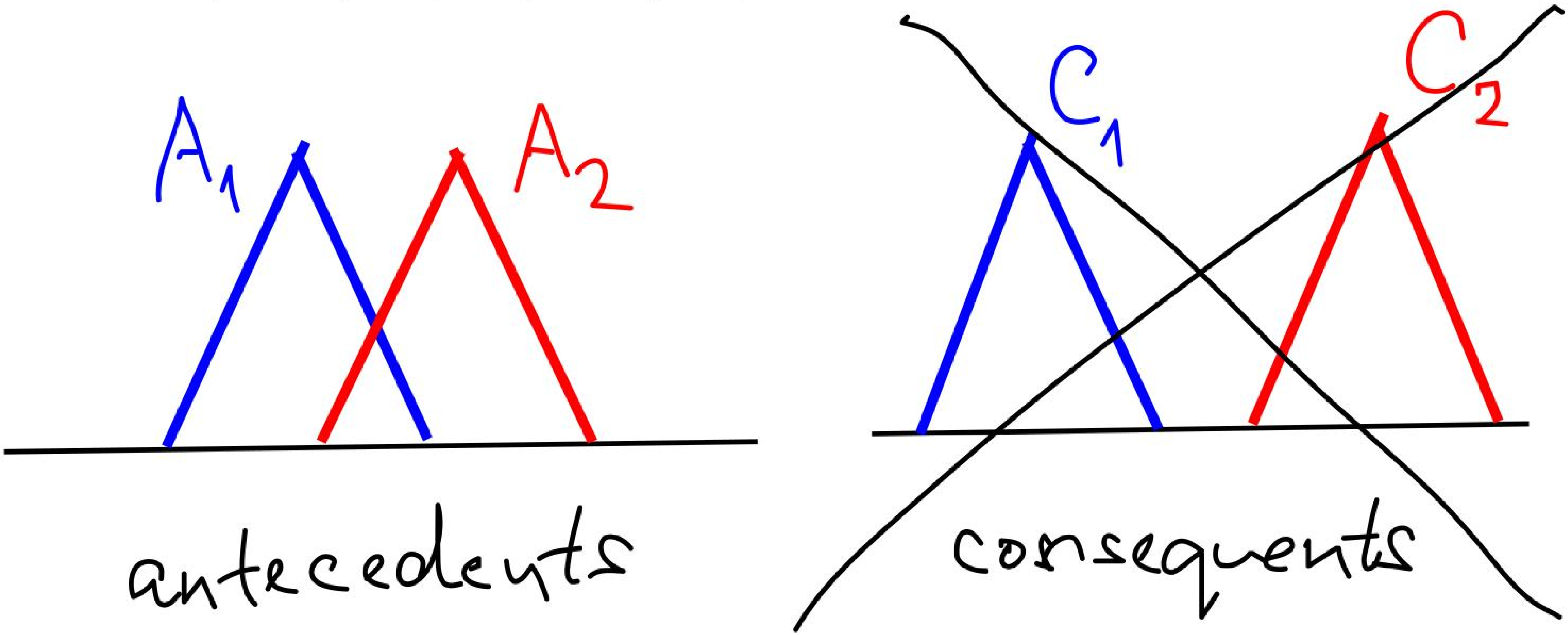

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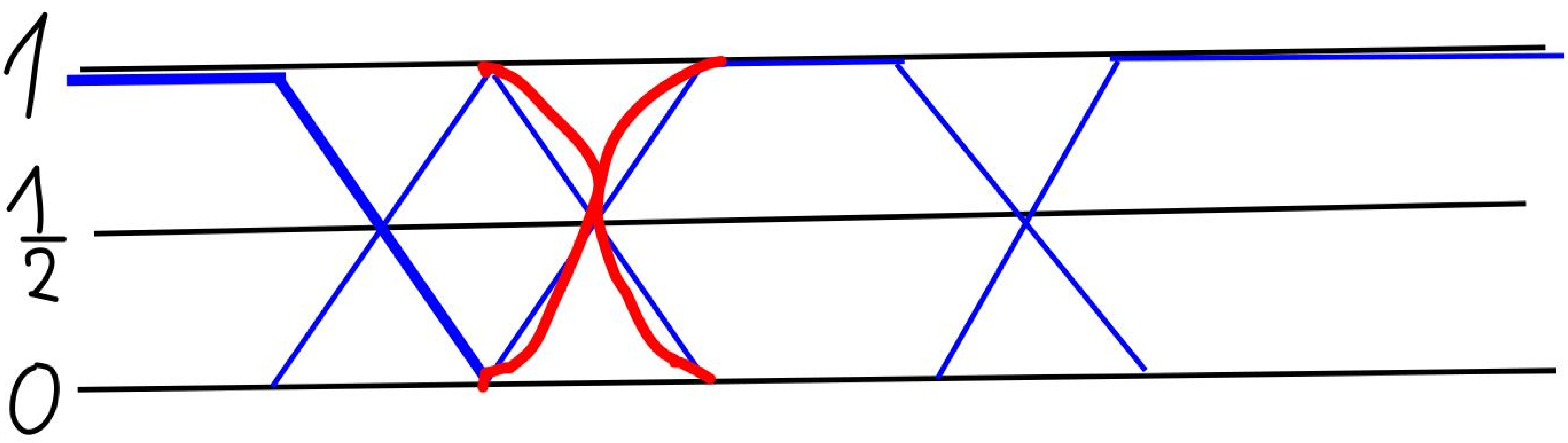

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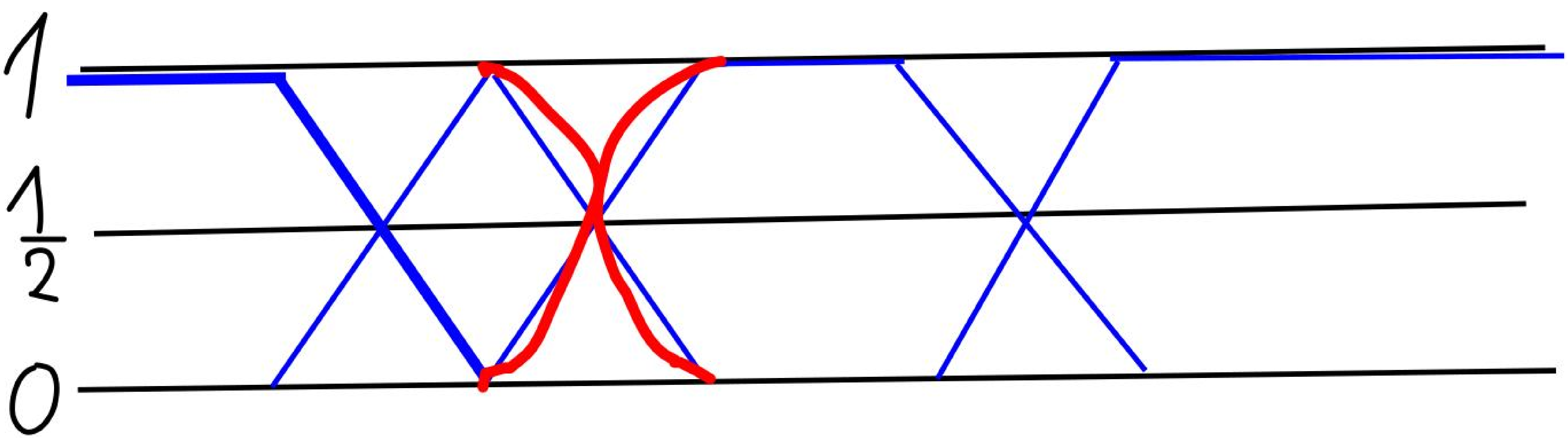

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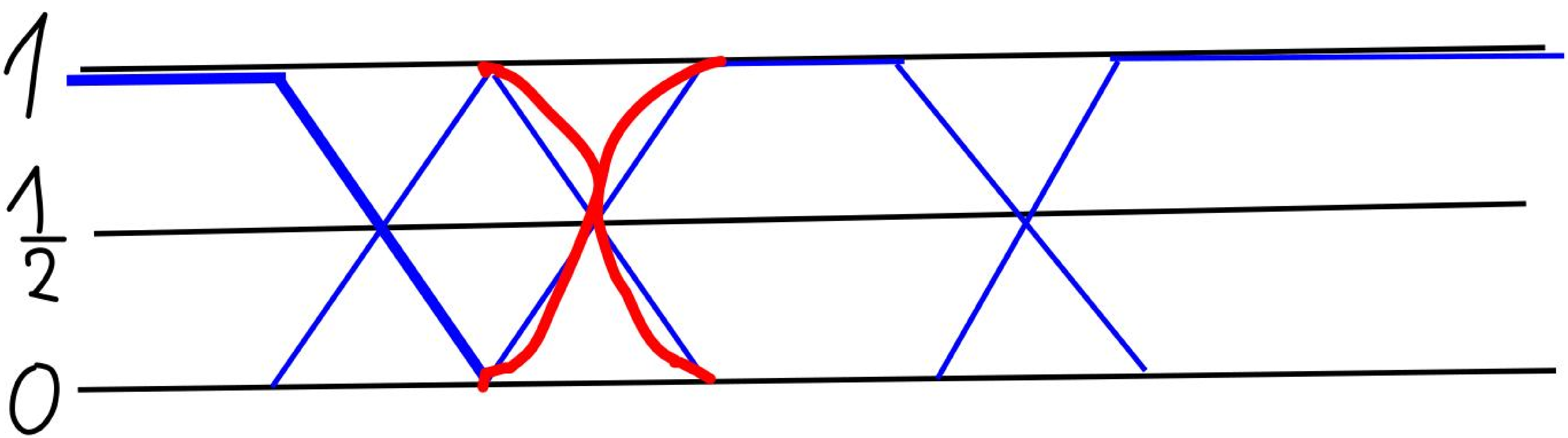

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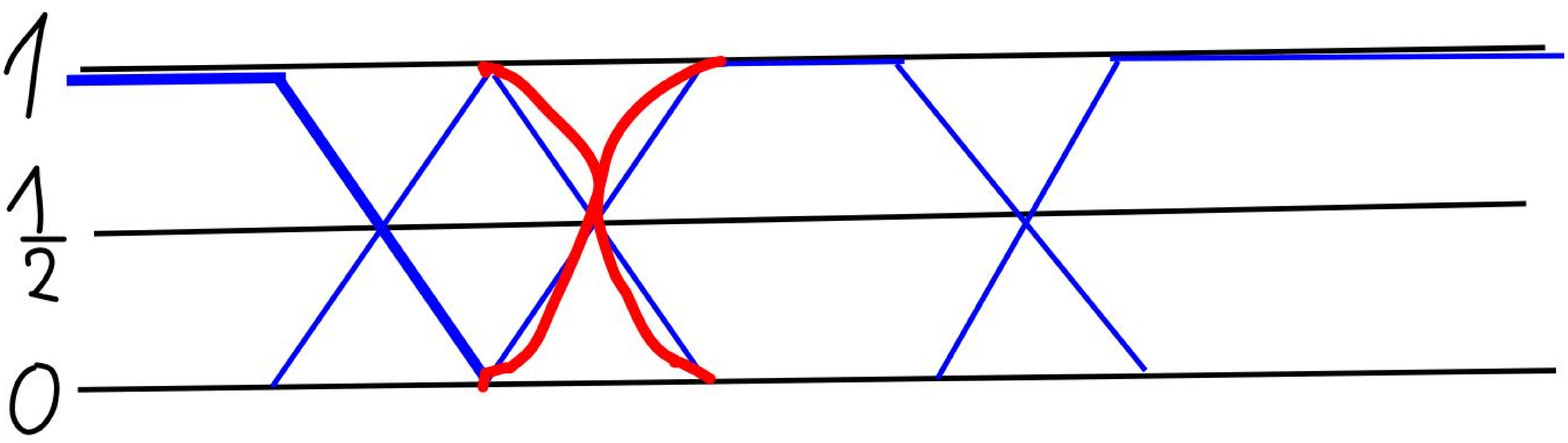

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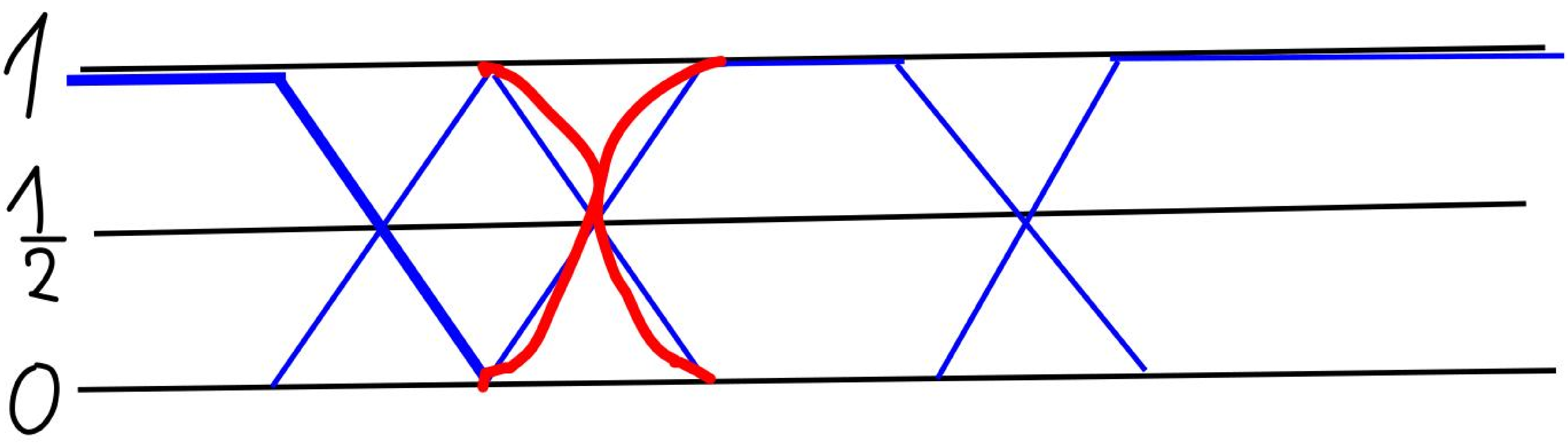

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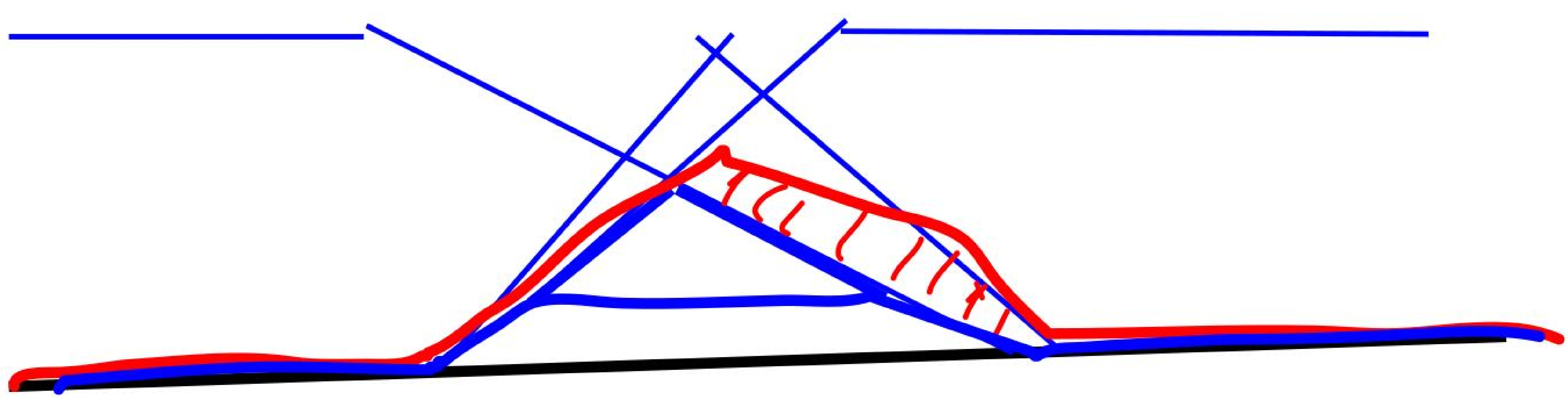

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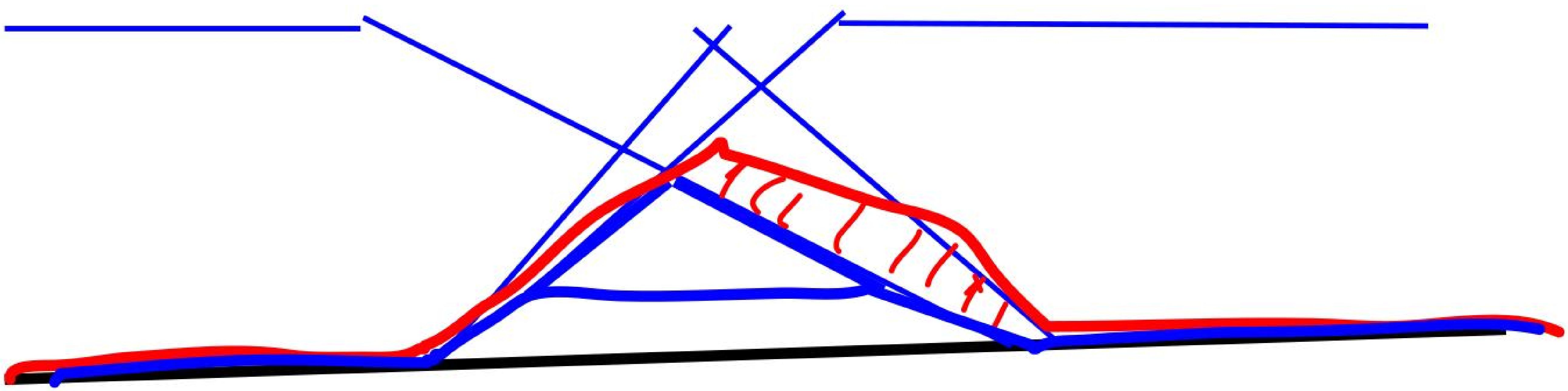

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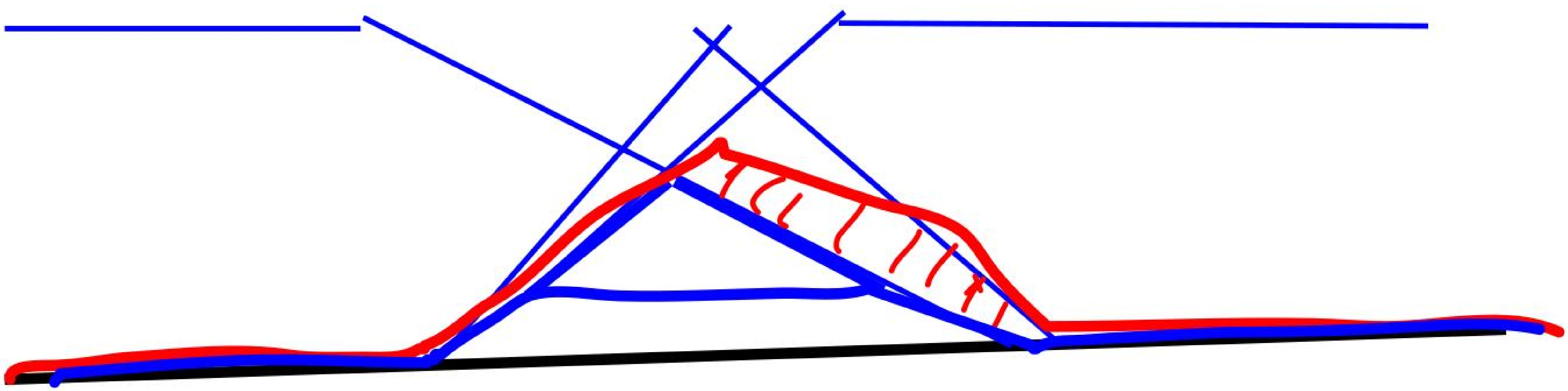

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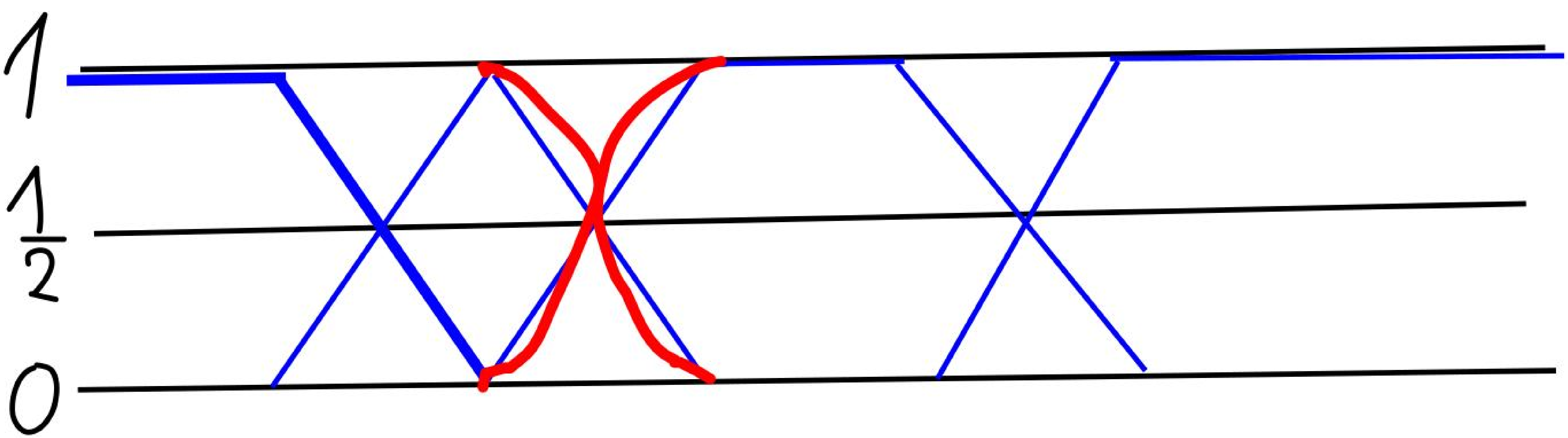

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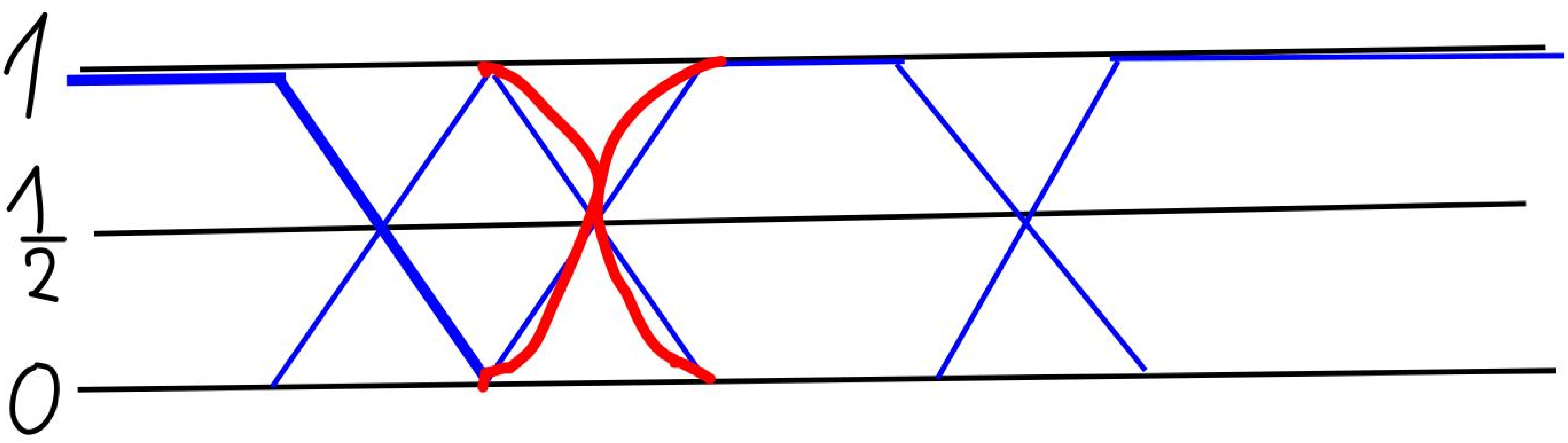

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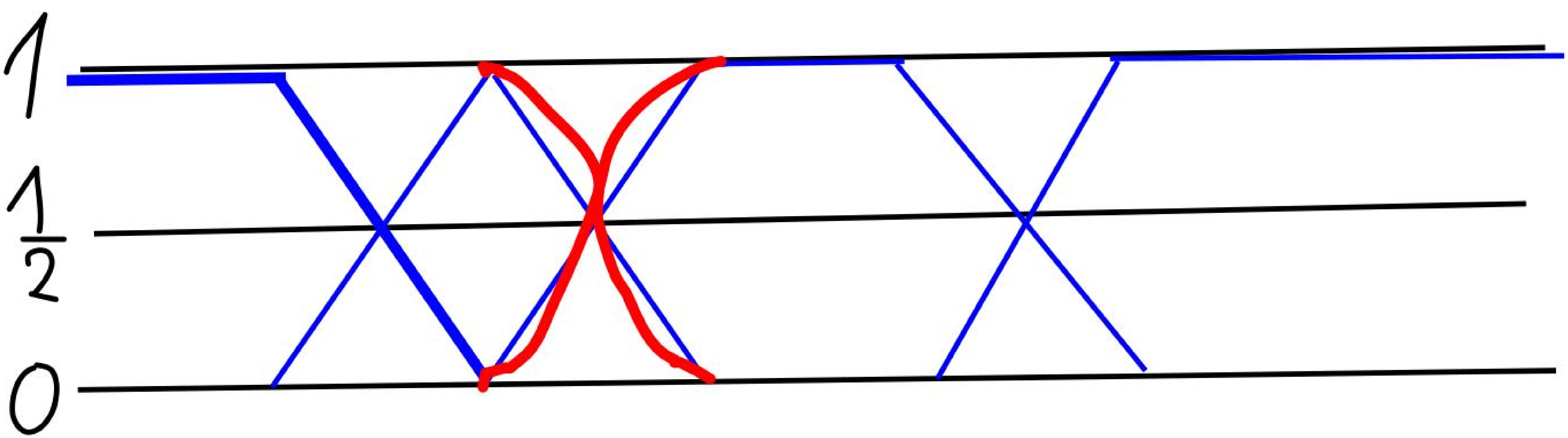

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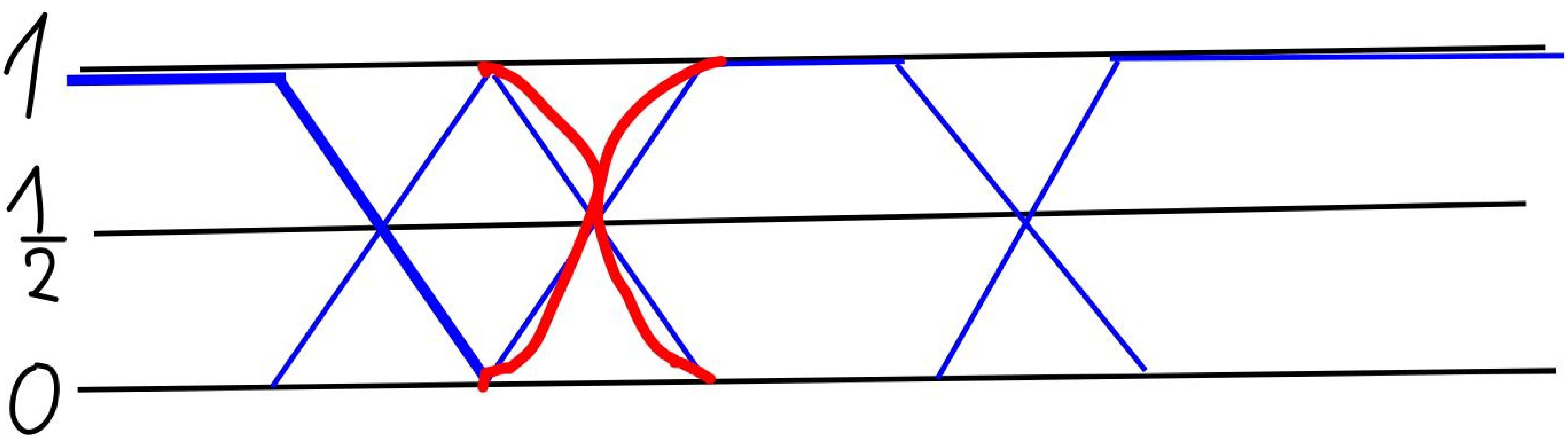

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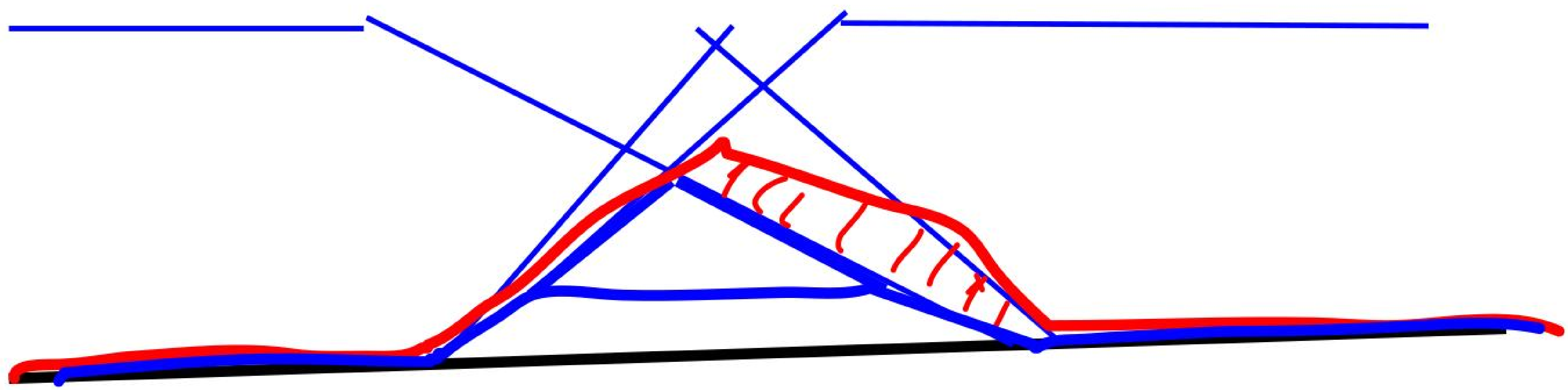

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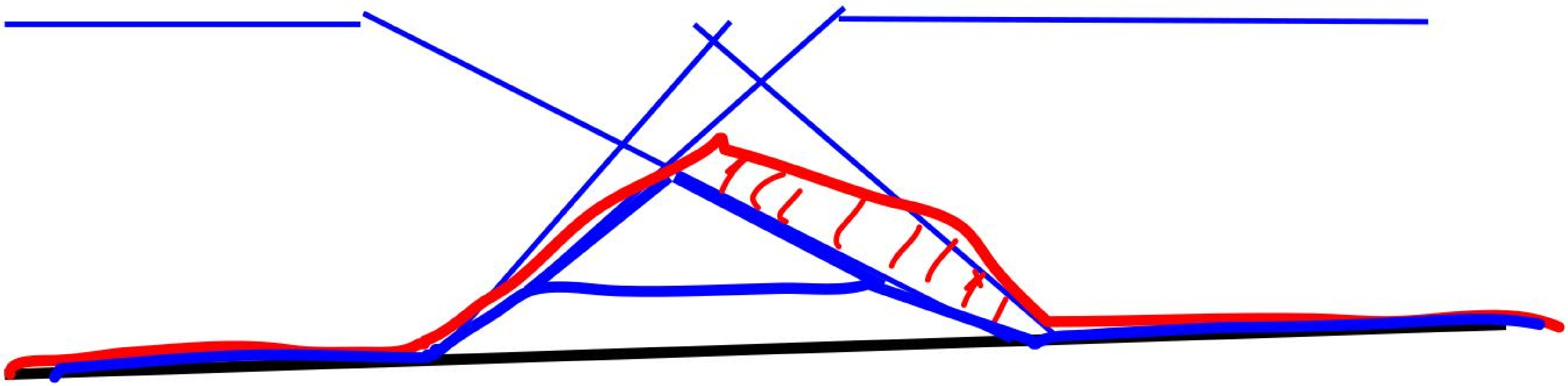

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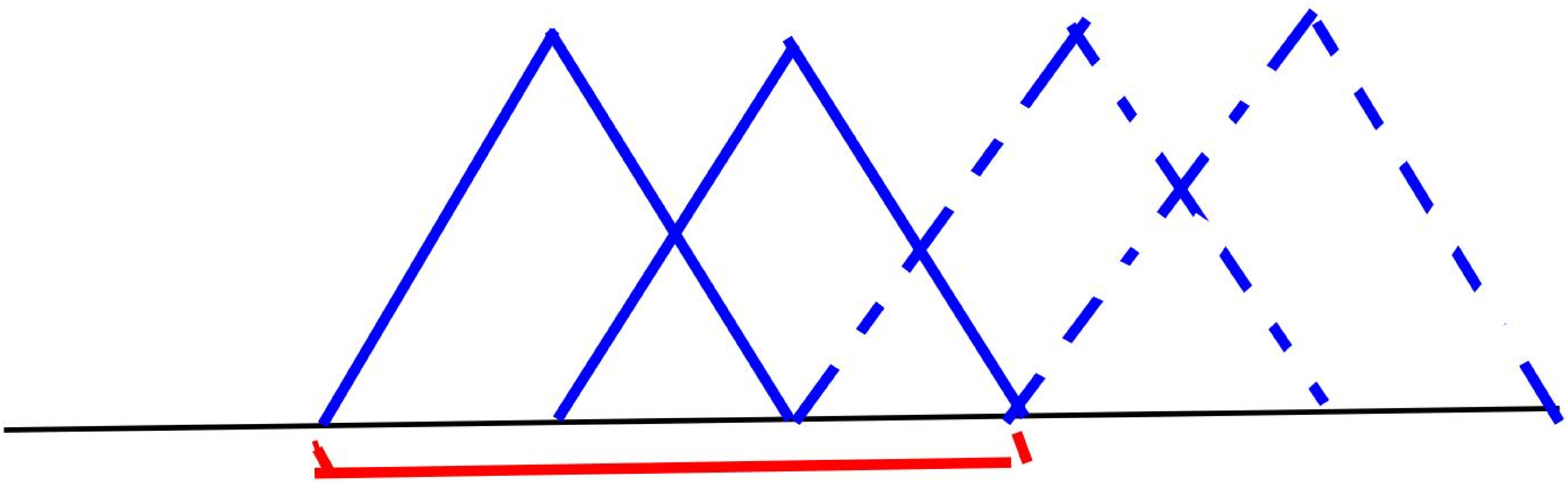

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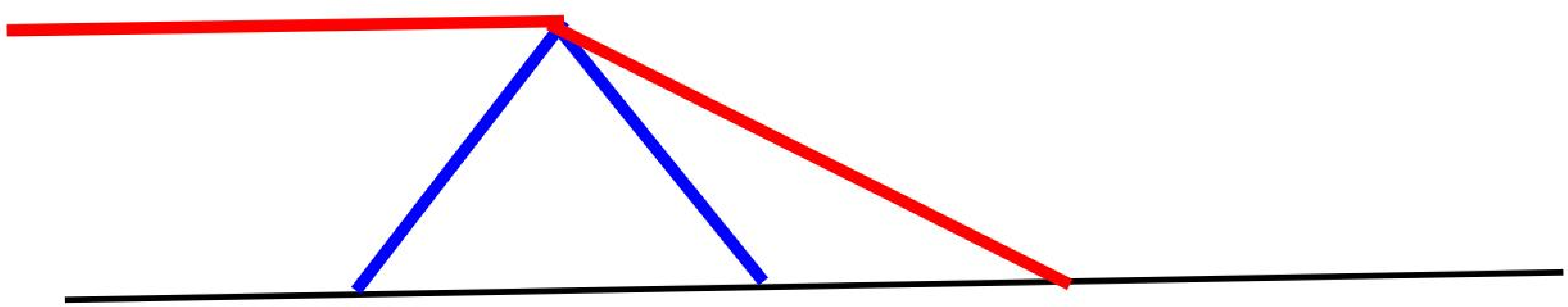

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Nevertheless, non-completeness is sometimes tolerated for the following reasons:

- In expert systems; “I don't know" could be a legitimate answer (of an expert system, not of a pilot!).
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The latter case can be formally described by an additional “**else rule**"
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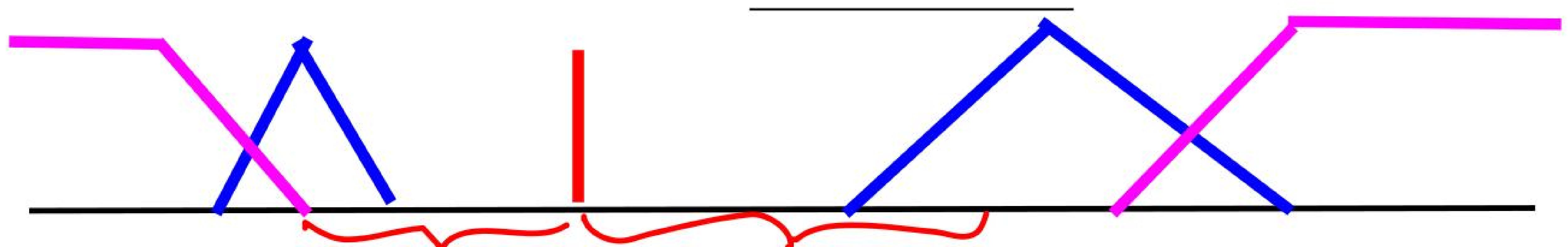

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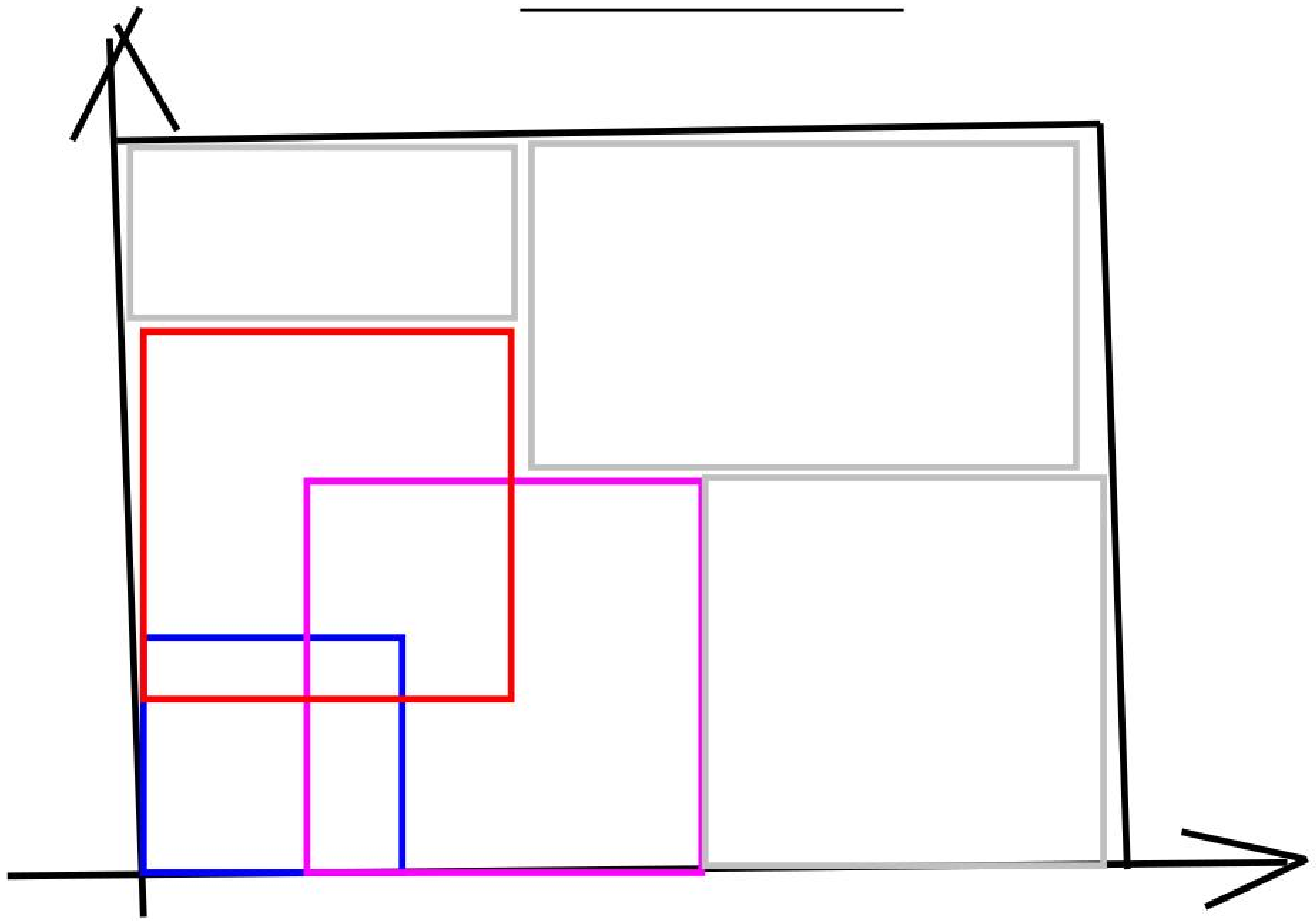

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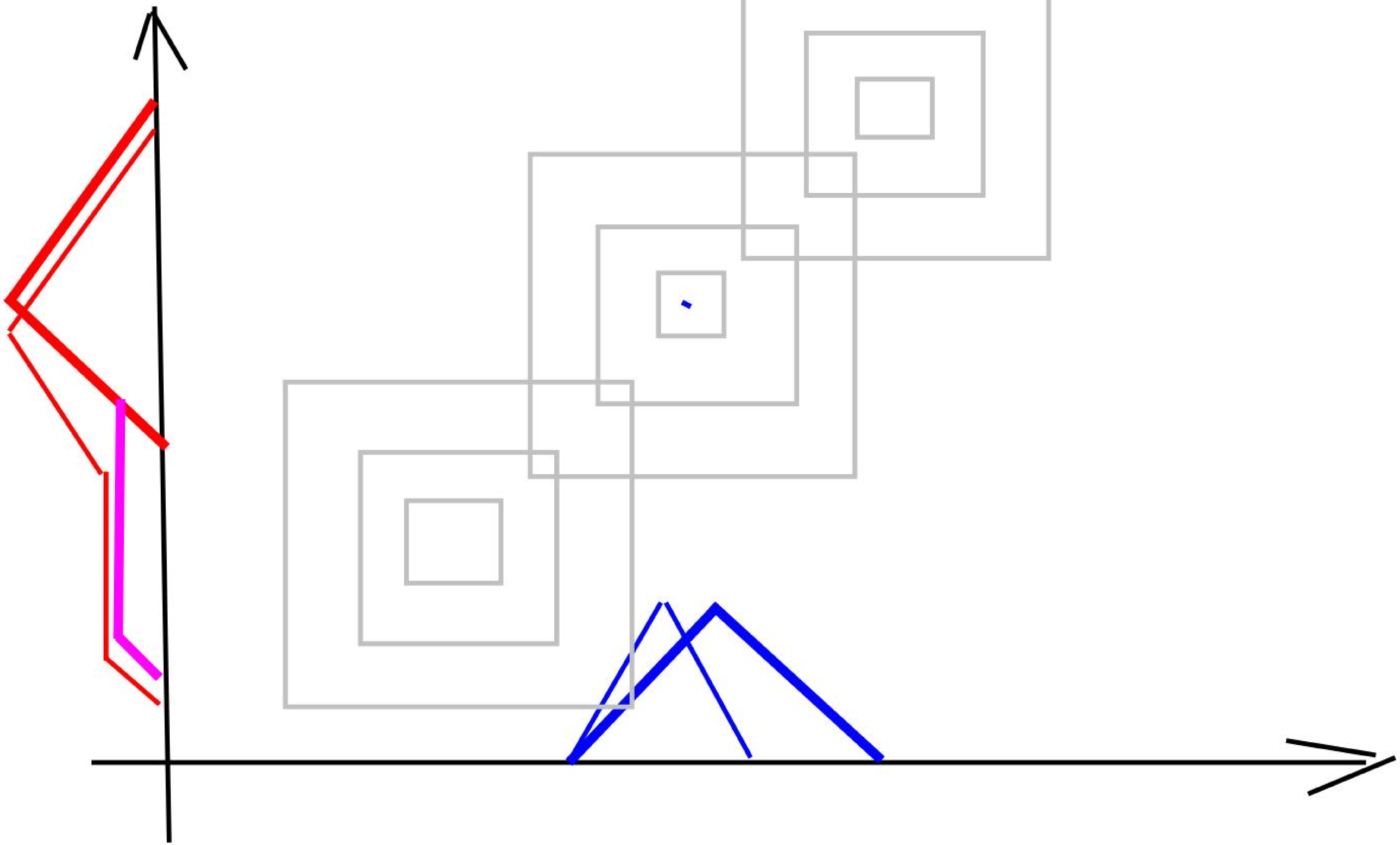

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For Mamdani–Assilian controller:

**Theorem**: $\forall j : \Phi_{\mathsf{MA}}(A_j) \geq C_j$ .

**Proof**: $X := A_j$ ,
$\mathcal{D}(X, A_j) = \mathcal{D}(A_j, A_j) = 1$  (due to normality),
$\Phi_{\mathsf{MA}}(A_j)(y) = \max\limits_i (\mathcal{D}(A_j, A_i) \underset{\cdot}{\wedge} C_i(y)) \geq \underbrace{\mathcal{D}(A_j, A_j)}_{1} \underset{\cdot}{\wedge} C_j(y) = C_j(y)$ .

# Correctness of Mamdani–Assilian controller



**Theorem** [de Baets 1996, Perfilieva, Tonis 1997]: $\big(\forall j : \Phi_{\mathsf{MA}}(A_j) = C_j\big)$ iff $\big(\forall i\ \forall j : \mathcal{D}(A_i, A_j) \le \mathcal{I}(C_i, C_j)\big)$,
where $\mathcal{I}(C_i, C_j) = \inf\limits_y \big(C_i(y) \underset{\cdot}{\rightarrow} C_j(y)\big)$
(the implication $\underset{\cdot}{\rightarrow}$ has to be the residuum of $\underset{\cdot}{\wedge}$).

Instead of $I(C_i, C_j)$ we may use $\mathcal{E}(C_i, C_j) = \inf\limits_y \big(C_i(y) \underset{\cdot}{\leftrightarrow} C_j(y)\big)$
(**degree of indistinguishability (equality)**),
where $\alpha \underset{\cdot}{\leftrightarrow} \beta = \min(\alpha \underset{\cdot}{\rightarrow} \beta, \beta \underset{\cdot}{\rightarrow} \alpha) = (\alpha \underset{\cdot}{\rightarrow} \beta) \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\rightarrow} \alpha)$.

**Proof**: The negation of the left-hand side is

$$\begin{aligned}
\exists j\ \exists y : \Phi_{\mathsf{MA}}(A_j)(y) &> C_j(y)\,,\\
\exists j\ \exists y\ \exists x : A_j(x) \underset{\cdot}{\wedge} R_{\mathsf{MA}}(x, y) &> C_j(y)\,,\\
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**Theorem** [de Baets 1996, Perfilieva, Tonis 1997]: $\big(\forall j : \Phi_{\mathsf{MA}}(A_j) = C_j\big)$ iff $\big(\forall i\ \forall j : \mathcal{D}(A_i, A_j) \le \mathcal{I}(C_i, C_j)\big)$,
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# Correctness of Mamdani–Assilian controller [Moser, Navara 1999]



If $\dot{\wedge}$ has no zero divisors (e.g., the minimum or product), then $\mathcal{D}(A_i, A_j) \leq \mathcal{E}(C_i, C_j)$ is satisfied in two situations:

- $\mathcal{E}(C_i, C_j) > 0$; then Supp $C_i =$ Supp $C_j$, which is rather unusual,
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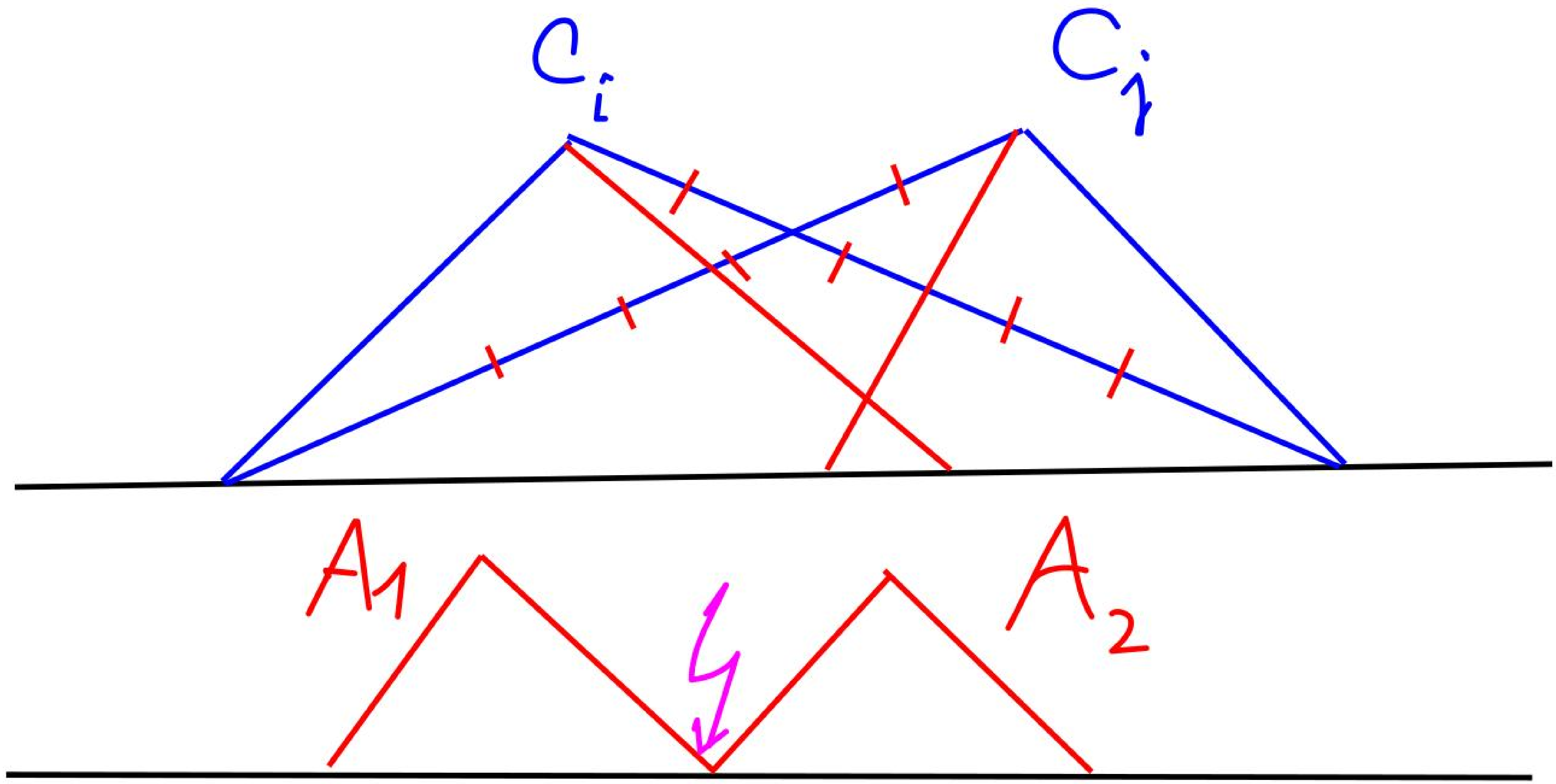

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# Correctness of residuum-based controller

**Theorem**: $\forall j : \Phi_{\mathsf{RES}}(A_j) \le C_j$ .

**Proof**: $X := A_j$ ,

$$\begin{aligned}\Phi_{\mathsf{RES}}(A_j)(y) &= \sup_x \big(A_j(x) \underset{\cdot}{\wedge} \min_i (A_i(x) \underset{\cdot}{\to} C_i(y))\big)\\ &\le \sup_x \big(A_j(x) \underset{\cdot}{\wedge} (A_j(x) \underset{\cdot}{\to} C_j(y))\big) \le C_j(y)\,.\end{aligned}$$

**Theorem**: If there is a fuzzy relation $R$ such that $\forall j : A_j \underset{\cdot}{\circ} R = C_j$ , then also $R_{\mathsf{RES}}$ satisfies these equalities (and it is the largest solution).

**Proof**: $\forall j\ \forall x\ \forall y :$

$$\begin{aligned}A_j(x) \underset{\cdot}{\wedge} R(x,y) &\le C_j(y)\\ R(x,y) &\le A_j(x) \underset{\cdot}{\to} C_j(y)\\ R(x,y) &\le \min_i \big(A_i(x) \underset{\cdot}{\to} C_i(y)\big) = R_{\mathsf{RES}}(x,y)\,,\end{aligned}$$

$$C_j = A_j \underset{\cdot}{\circ} R \le A_j \underset{\cdot}{\circ} R_{\mathsf{RES}} \le C_j\,.$$

# Correctness of Mamdani–Assilian controller [Moser, Navara 1999]



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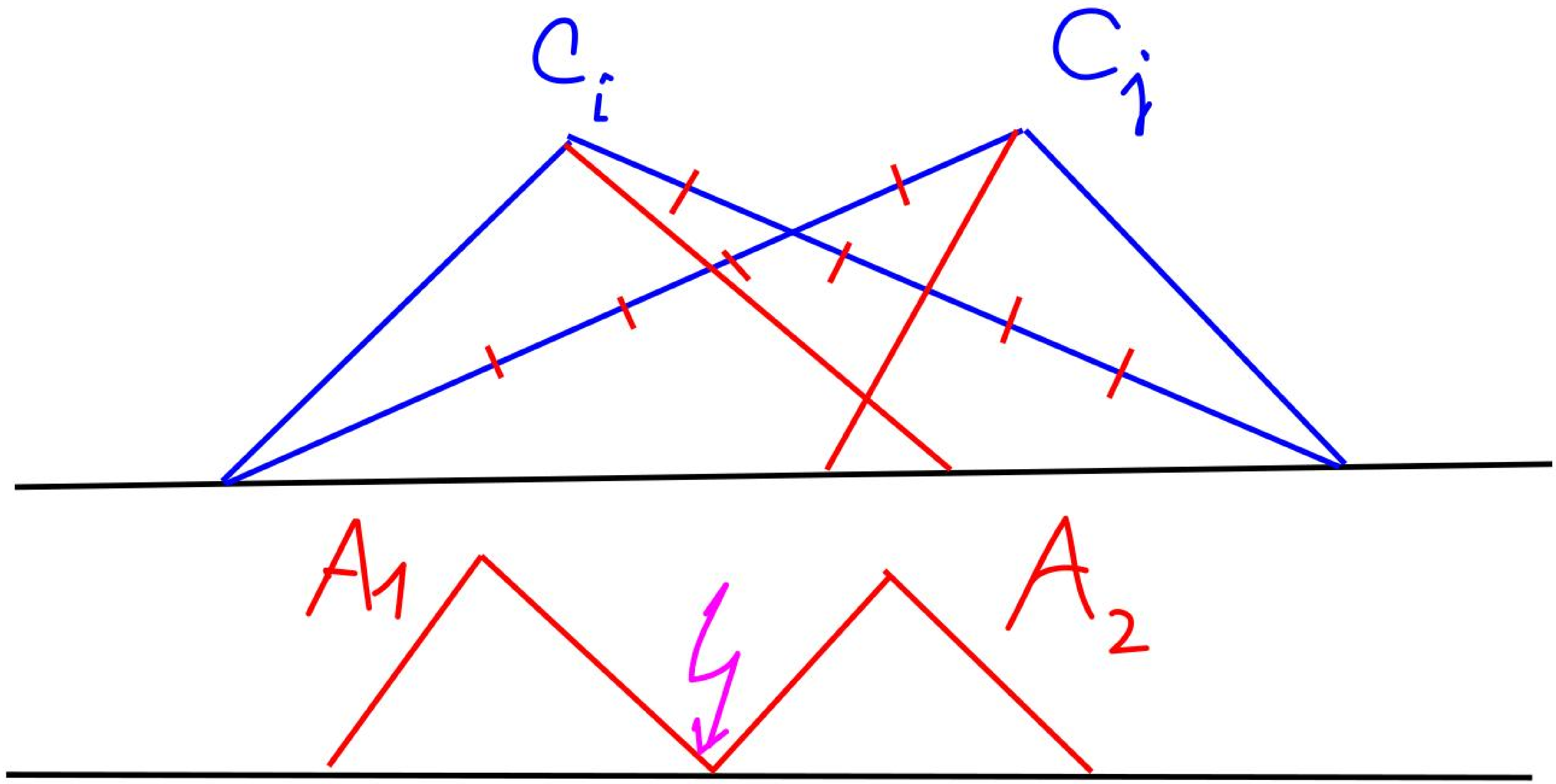

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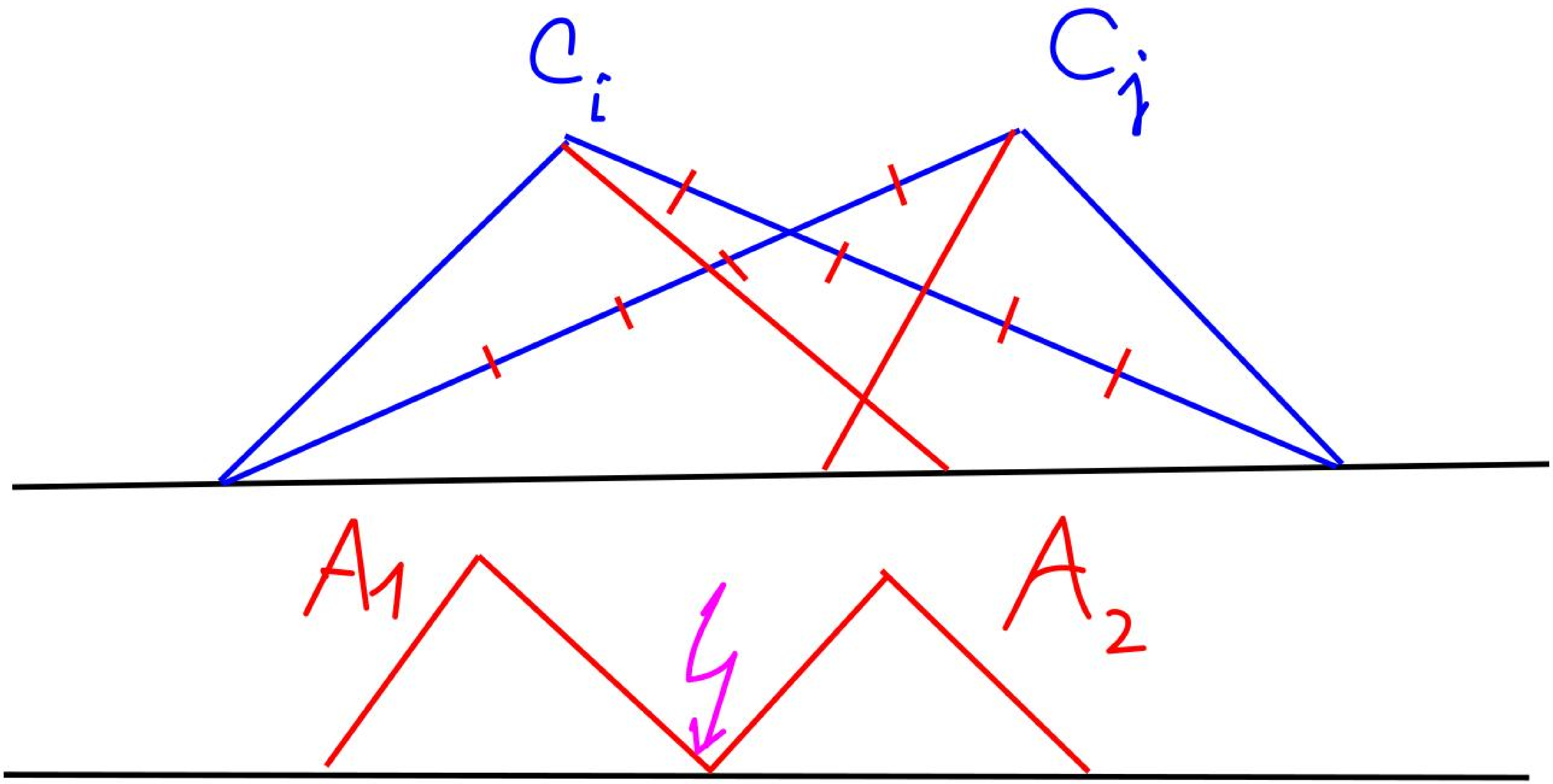

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**Theorem** [de Baets 1996, Perfilieva, Tonis 1997]: $\left(\forall j : \Phi_{\mathsf{MA}}(A_j) = C_j\right)$ iff $\left(\forall i\ \forall j : \mathcal{D}(A_i, A_j) \le \mathcal{I}(C_i, C_j)\right)$,
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When $\forall j : \Phi(A_j) = A_j \underset{\cdot}{\circ} R_{\mathsf{MA}} = C_j$ ? (A system of fuzzy relational equations for a fuzzy relation $R_{\mathsf{MA}}$.)

For Mamdani–Assilian controller:

**Theorem**: $\forall j : \Phi_{\mathsf{MA}}(A_j) \geq C_j$ .

**Proof**: $X := A_j$ ,
$\mathcal{D}(X, A_j) = \mathcal{D}(A_j, A_j) = 1$ (due to normality),
$\Phi_{\mathsf{MA}}(A_j)(y) = \max\limits_i (\mathcal{D}(A_j, A_i) \underset{\cdot}{\wedge} C_i(y)) \geq \underbrace{\mathcal{D}(A_j, A_j)}_{1} \underset{\cdot}{\wedge} C_j(y) = C_j(y)$ .

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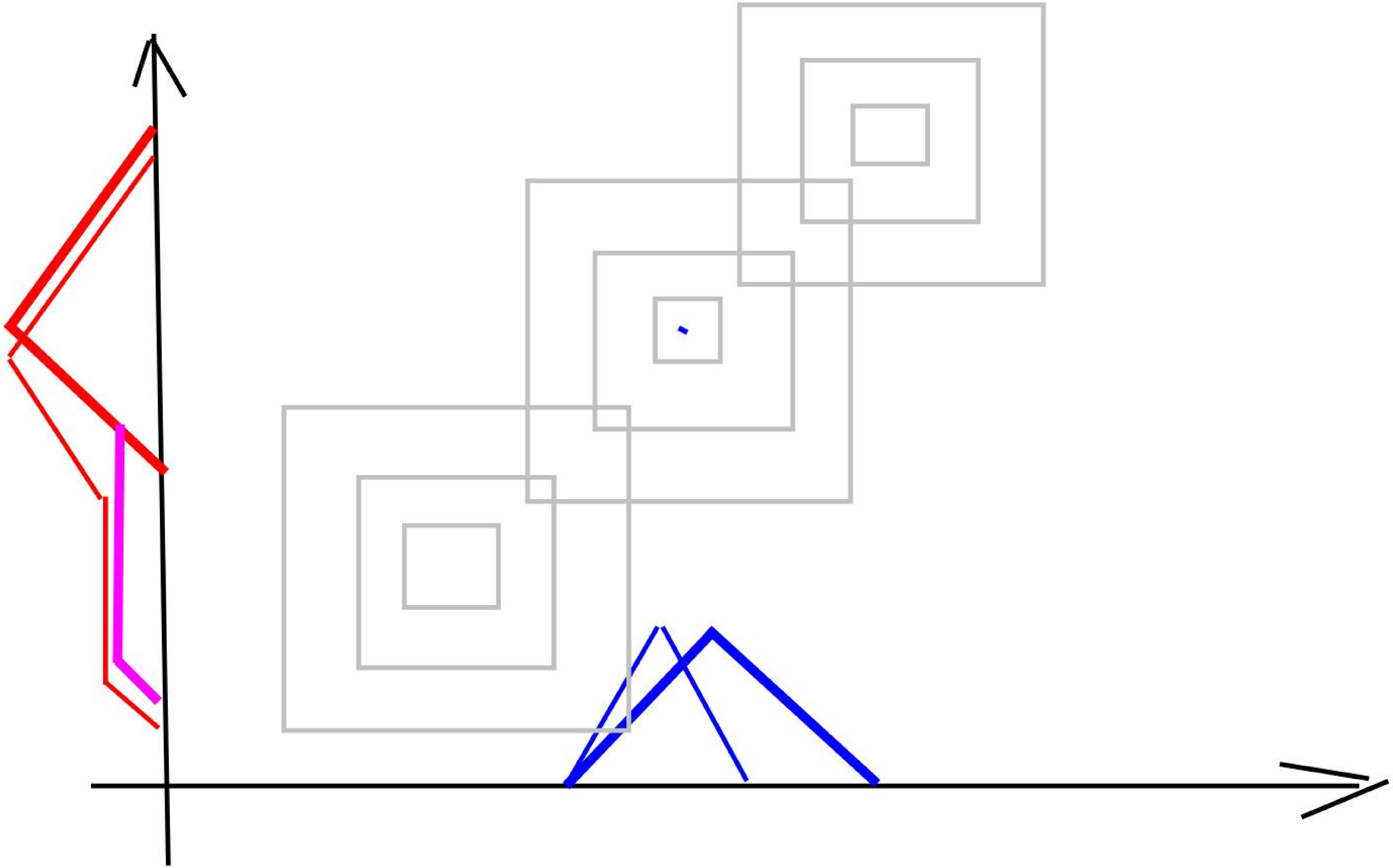

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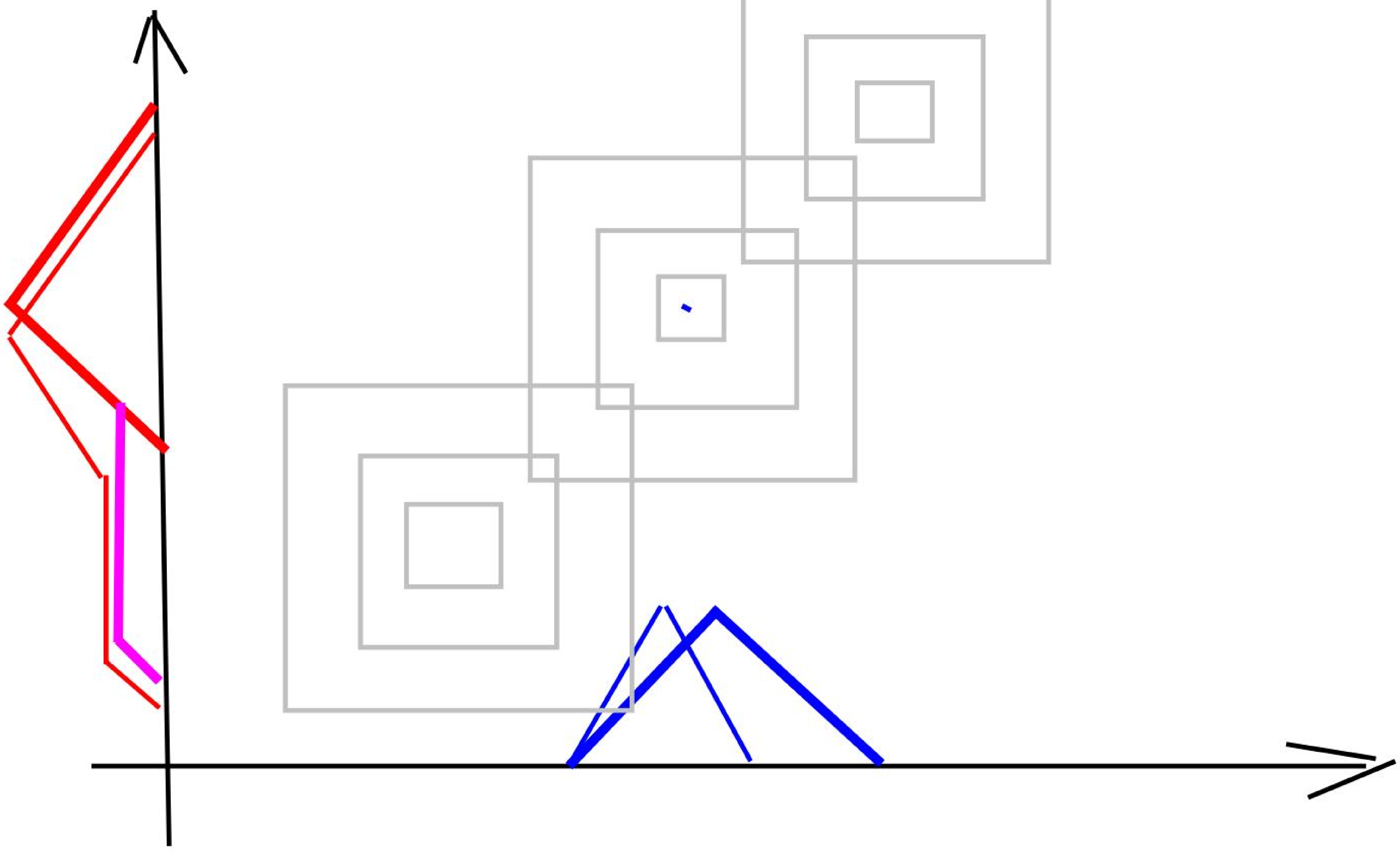
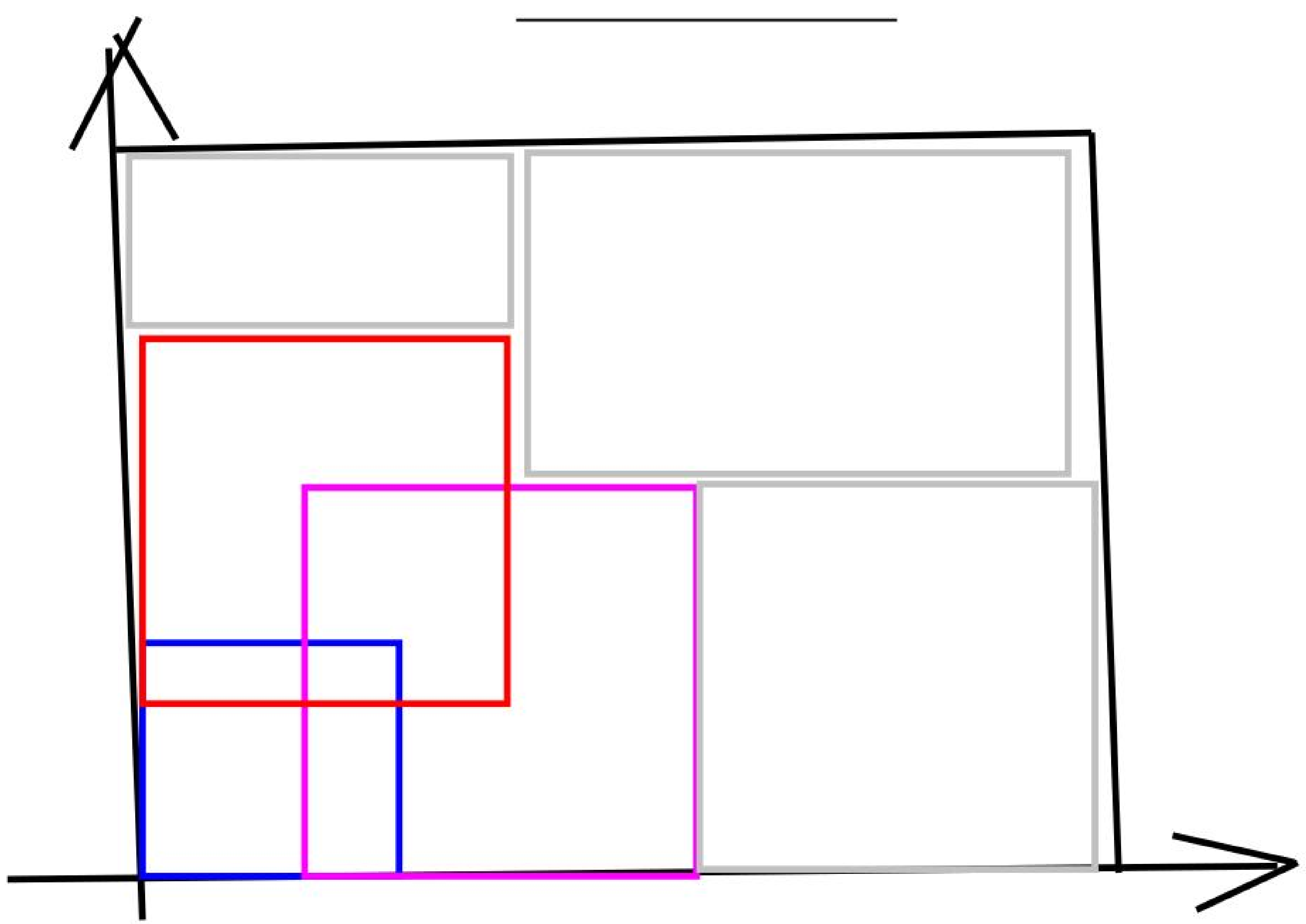


# Completeness of the rule base



Omitting rules for some situations is motivated by the attempt to reduce the number of rules (**curse of dimensionality**).

Sometimes the table of linguistic rules does not cover some combinations of linguistic variables.
This does not obviously mean that the antecedents are not complete; the case may be covered by neighbouring rules, although with a smaller degree of firing.



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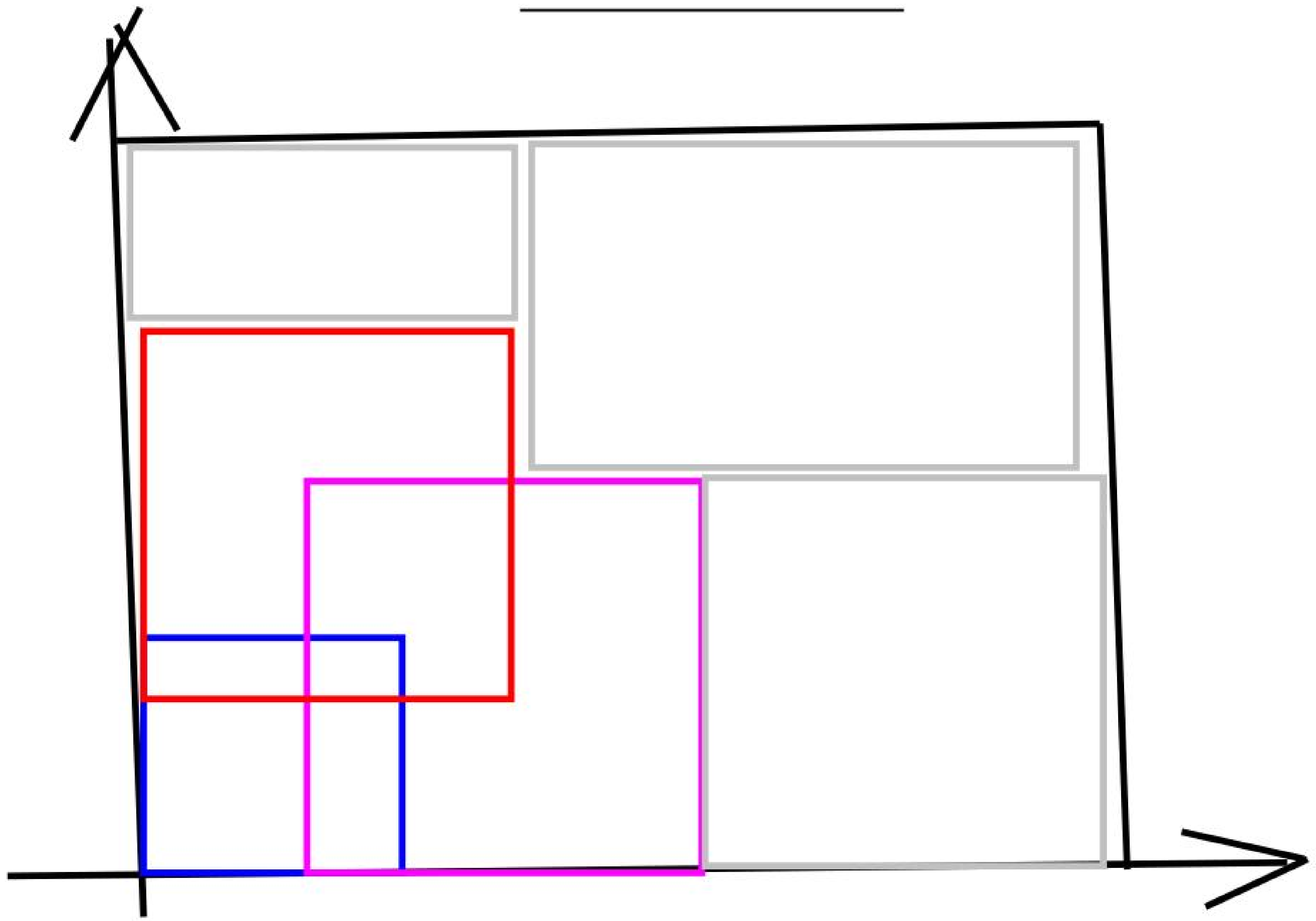

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Completeness is required, because in any situation we need at least one firing rule.

Nevertheless, non-completeness is sometimes tolerated for the following reasons:

- In expert systems; "I don't know" could be a legitimate answer (of an expert system, not of a pilot!).
- The input is impossible (then do not include it in the input space!).
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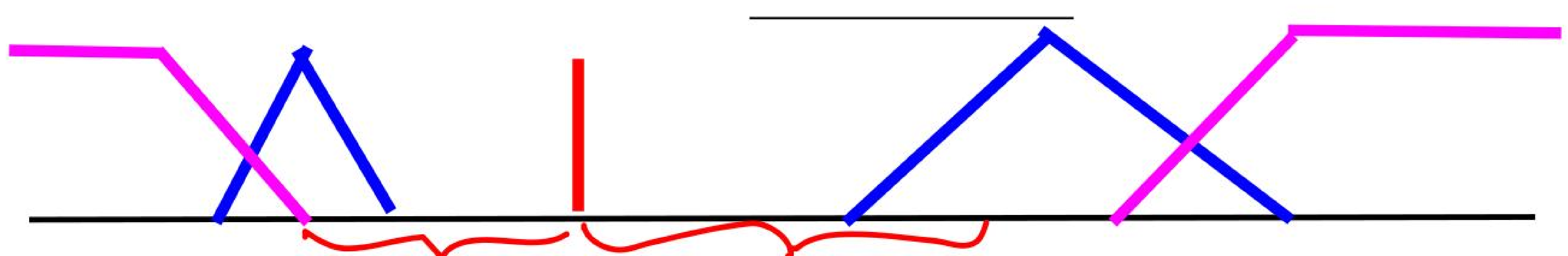

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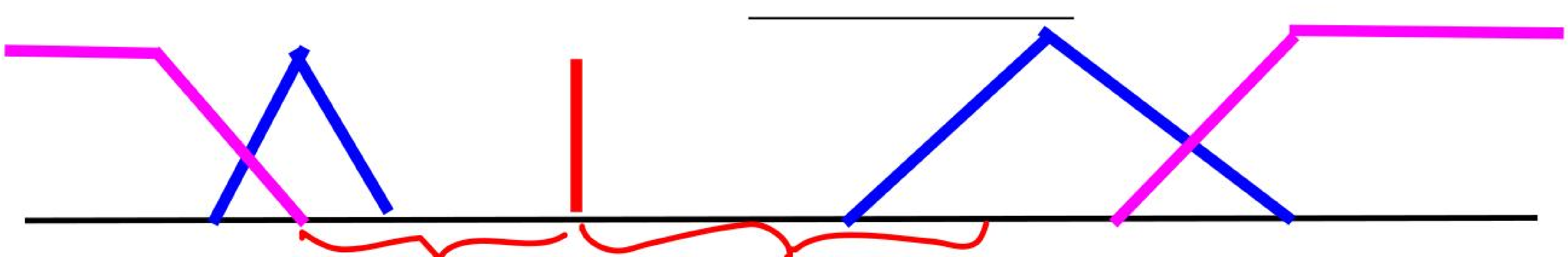

## Requirements on the rule base [Moser, Navara 2002]

- **Local correctness** (**interaction**): $\forall j : \Phi(A_j) = C_j$ .
- **Strong completeness**: $\forall$ normal $X \in \mathcal{F}(\mathcal{X}) : \Phi(X) \not\subseteq \bigcap_i C_i$, where the fuzzy intersection is standard (computed using $\min$).
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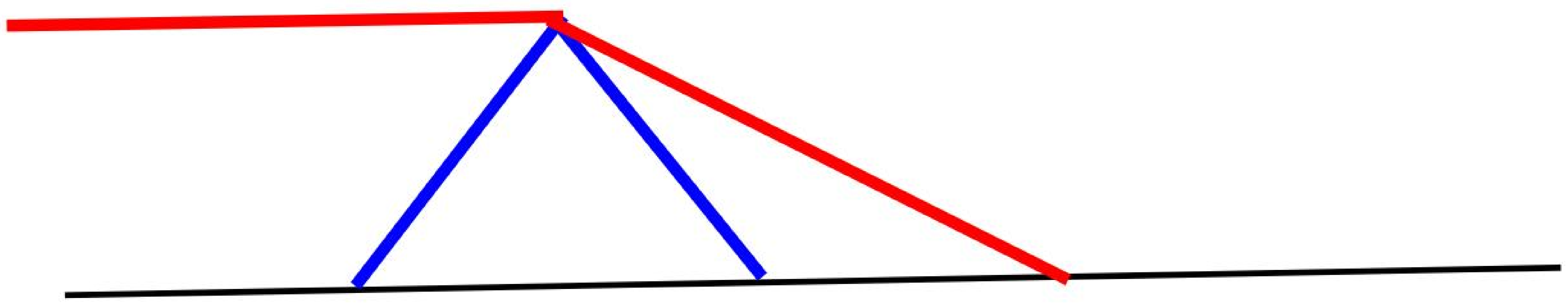

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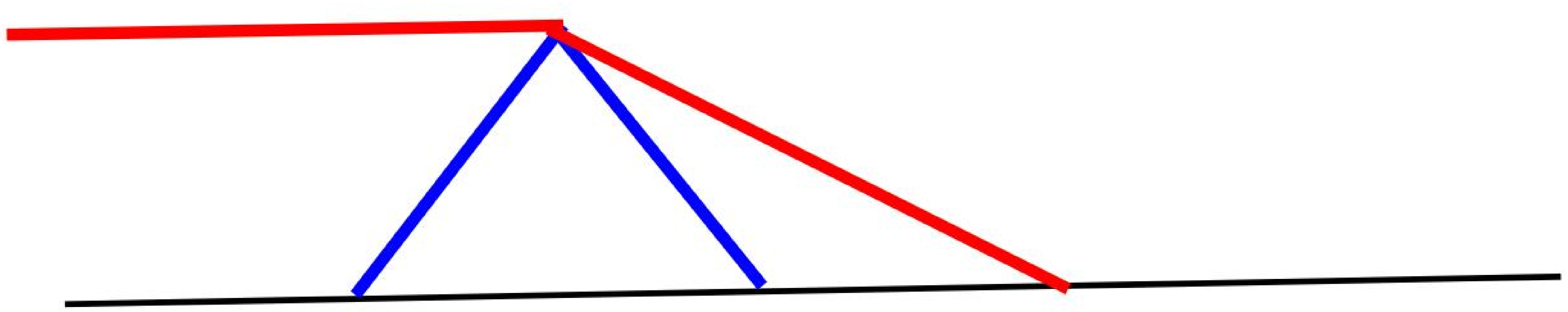

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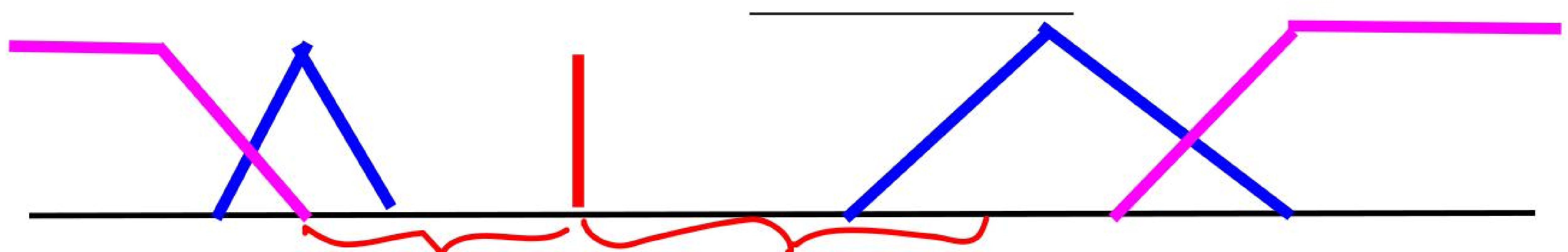

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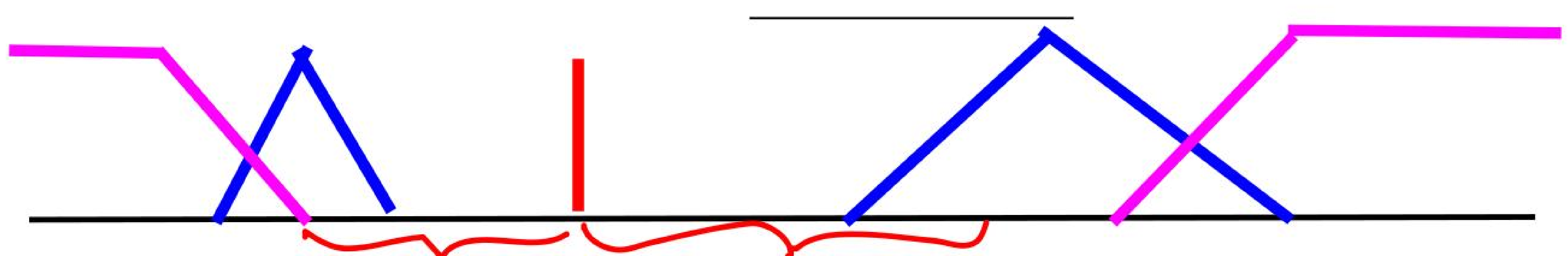

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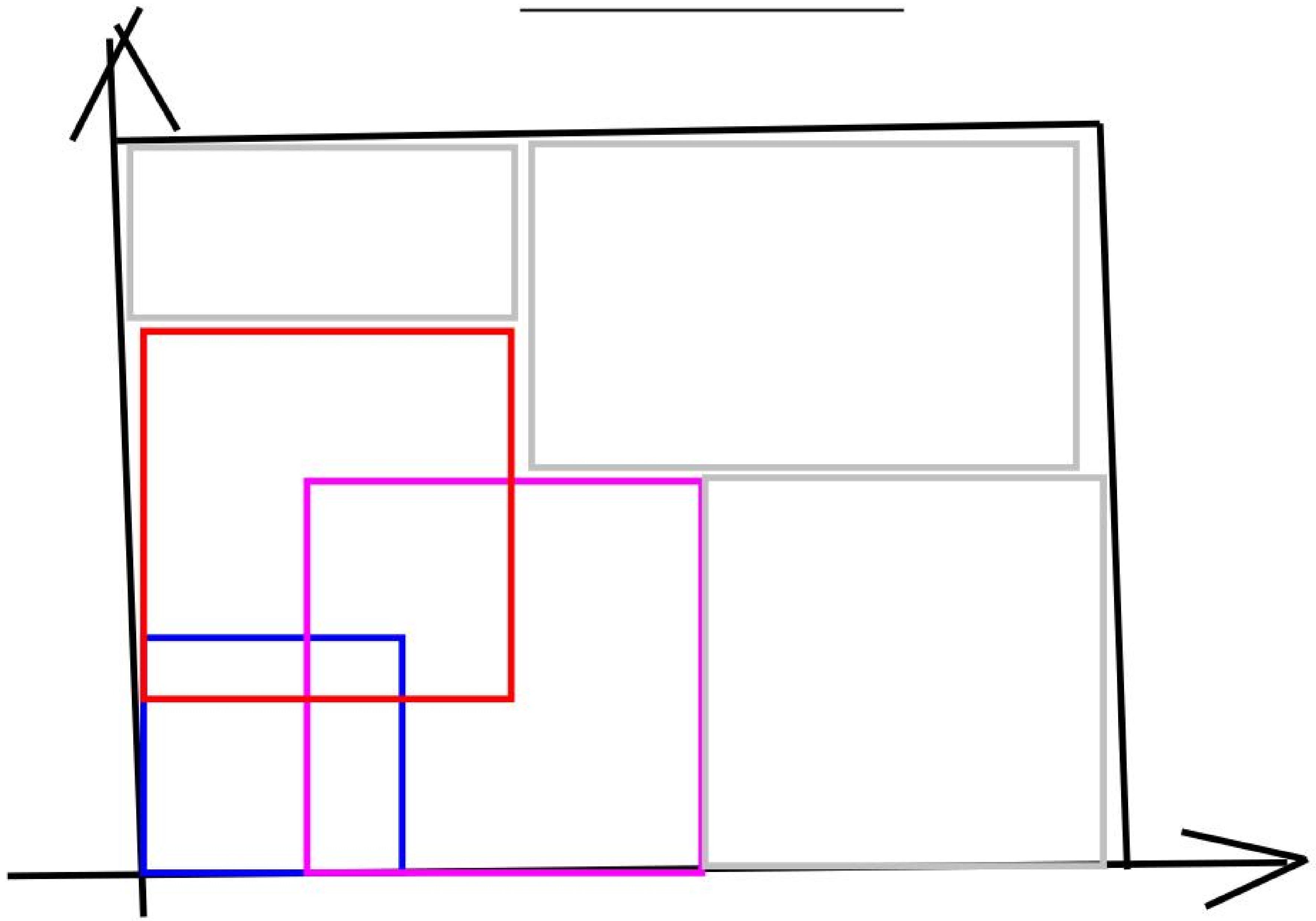

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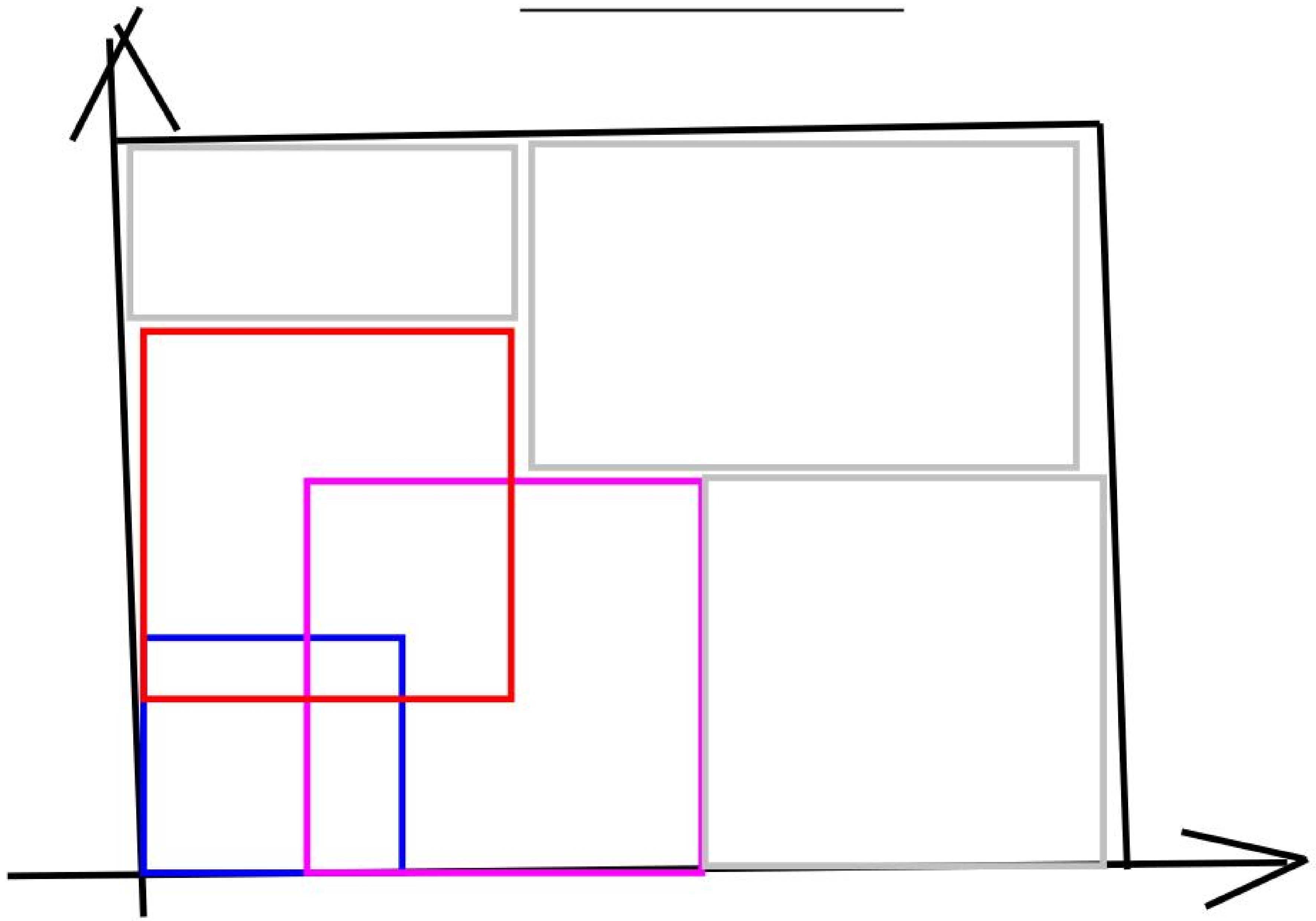

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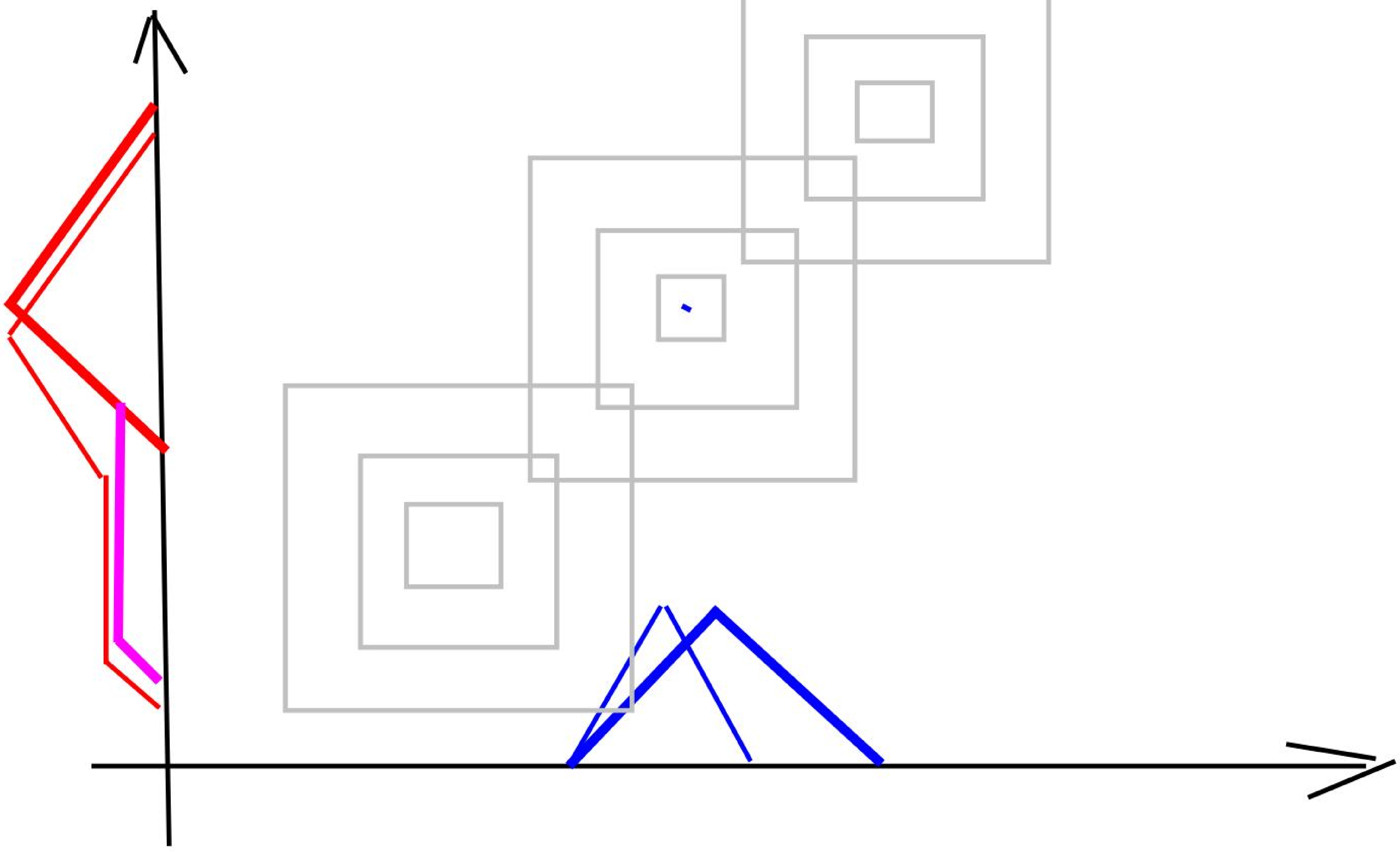

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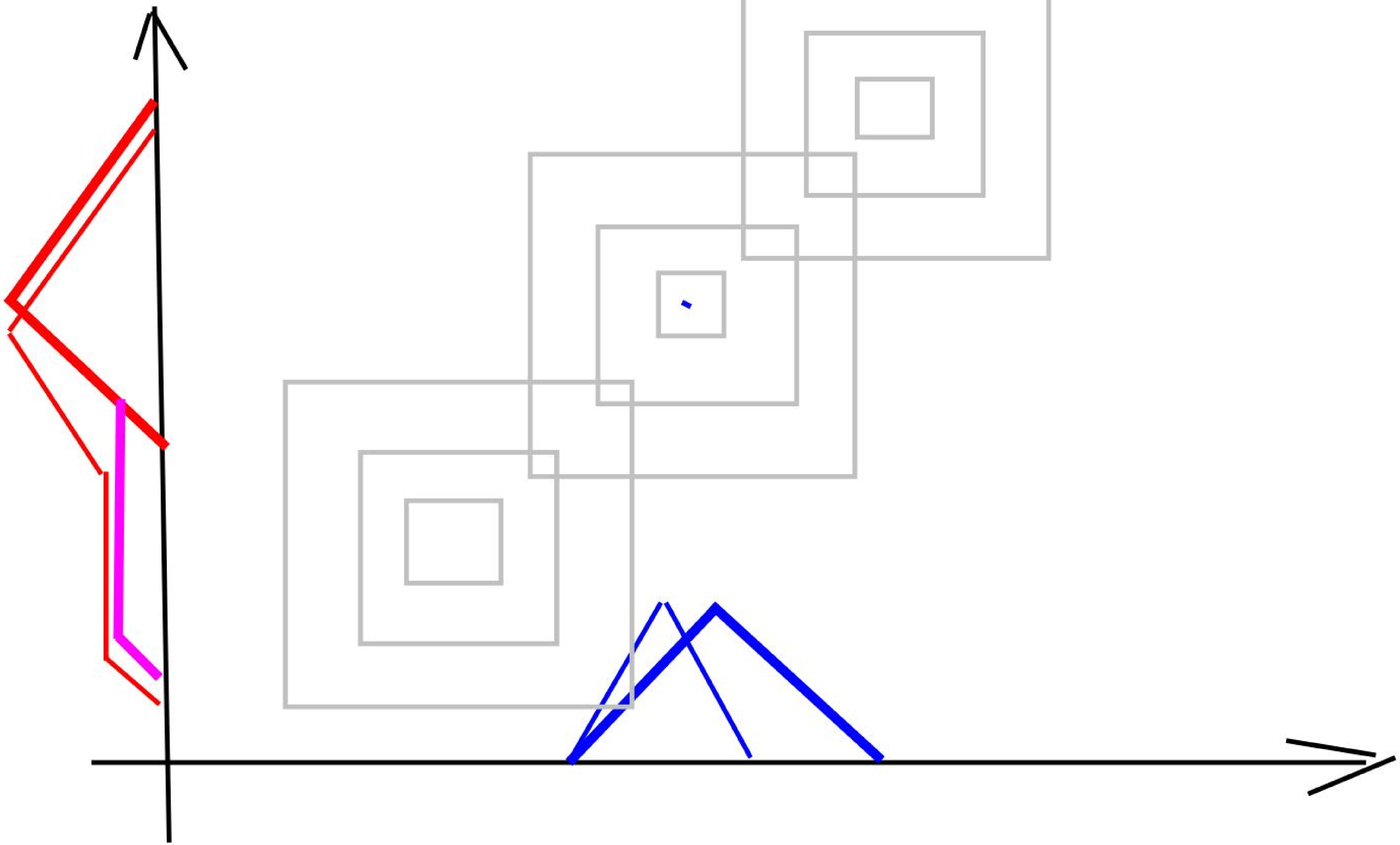

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When $\forall j : \Phi(A_j) = A_j \underset{\cdot}{\circ} R_{\mathsf{MA}} = C_j$ ? (A system of fuzzy relational equations for a fuzzy relation $R_{\mathsf{MA}}$.)

For Mamdani–Assilian controller:

**Theorem**: $\forall j : \Phi_{\mathsf{MA}}(A_j) \geq C_j$ .

**Proof**: $X := A_j$ ,
$\mathcal{D}(X, A_j) = \mathcal{D}(A_j, A_j) = 1$ (due to normality),
$\Phi_{\mathsf{MA}}(A_j)(y) = \max\limits_i (\mathcal{D}(A_j, A_i) \underset{\cdot}{\wedge} C_i(y)) \geq \underbrace{\mathcal{D}(A_j, A_j)}_{1} \underset{\cdot}{\wedge} C_j(y) = C_j(y)$ .

# Correctness of Mamdani–Assilian controller

**Theorem** [de Baets 1996, Perfilieva, Tonis 1997]: $\big(\forall j : \Phi_{\mathsf{MA}}(A_j) = C_j\big)$ iff
$\big(\forall i\ \forall j : \mathcal{D}(A_i, A_j) < \mathcal{I}(C_i, C_j)\big)$,
where $\boxed{\mathcal{I}(C_i, C_j) = \inf\limits_y \big(C_i(y) \underset{\cdot}{\to} C_j(y)\big)}$
(the implication $\underset{\cdot}{\to}$ has to be the residuum of $\underset{\cdot}{\wedge}$).

Instead of $I(C_i, C_j)$ we may use $\boxed{\mathcal{E}(C_i, C_j) = \inf\limits_y \big(C_i(y) \underset{\cdot}{\leftrightarrow} C_j(y)\big)}$
(**degree of indistinguishability (equality)**),
where $\alpha \underset{\cdot}{\leftrightarrow} \beta = \min(\alpha \underset{\cdot}{\to} \beta, \beta \underset{\cdot}{\to} \alpha) = (\alpha \underset{\cdot}{\to} \beta) \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\to} \alpha)$.

**Proof**: The negation of the left-hand side is

$$
\begin{aligned}
\exists j\ \exists y : \Phi_{\mathsf{MA}}(A_j)(y) &> C_j(y)\,, \\
\exists j\ \exists y\ \exists x : A_j(x) \underset{\cdot}{\wedge} R_{\mathsf{MA}}(x, y) &> C_j(y)\,, \\
\exists i\ \exists j\ \exists y\ \exists x : A_j(x) \underset{\cdot}{\wedge} A_i(x) \underset{\cdot}{\wedge} C_i(y) &> C_j(y)\,, \\
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\end{aligned}
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which is the negation of the right-hand side.

# Correctness of Mamdani–Assilian controller

**Theorem** [de Baets 1996, Perfilieva, Tonis 1997]: $\big(\forall j : \Phi_{\mathsf{MA}}(A_j) = C_j\big)$ iff $\big(\forall i\ \forall j : \mathcal{D}(A_i, A_j) \le \mathcal{I}(C_i, C_j)\big)$,
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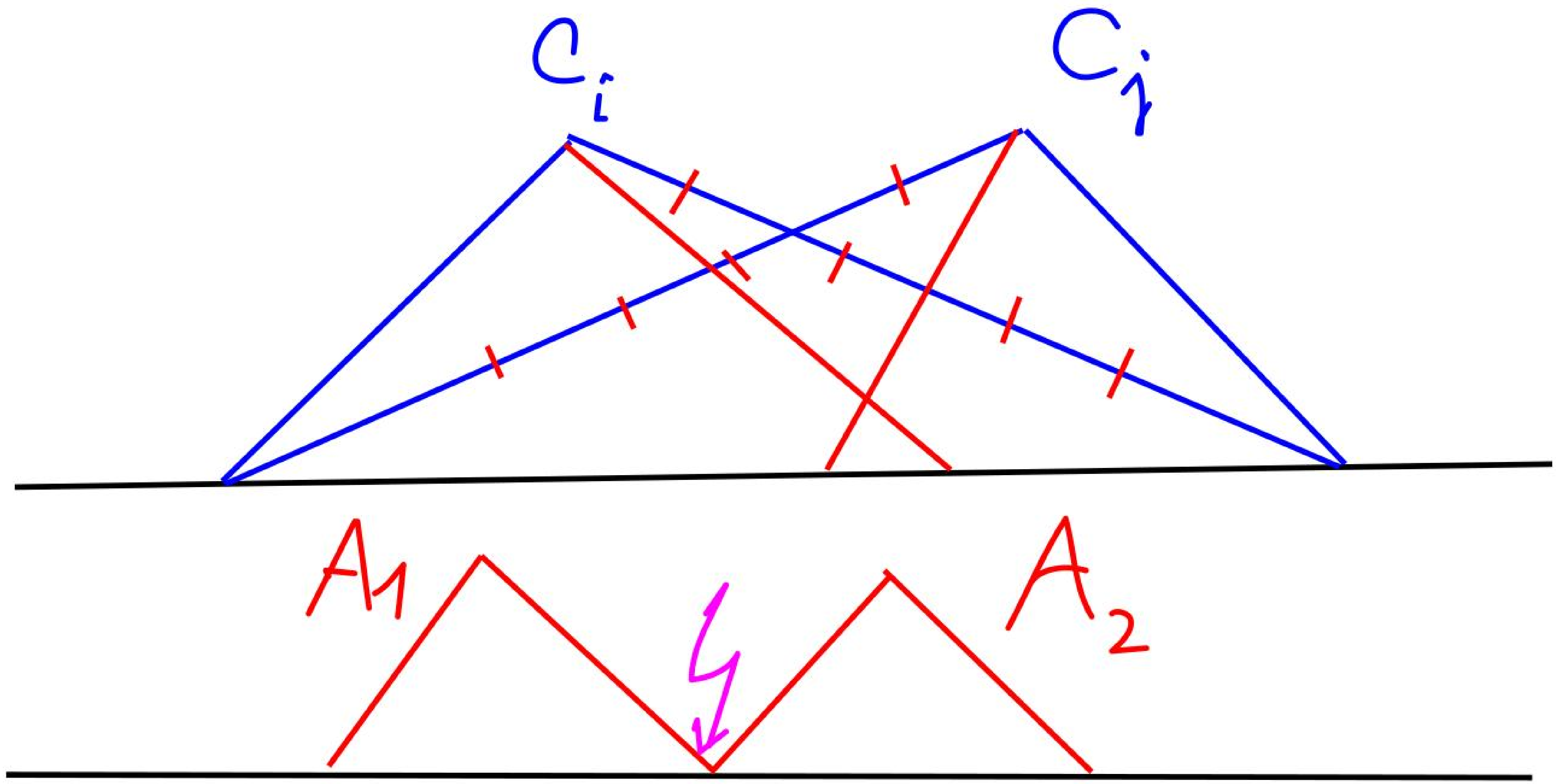
\end{aligned}$$

which is the negation of the right-hand side.

# Correctness of Mamdani–Assilian controller [Moser, Navara 1999]

If $\wedge$ has no zero divisors (e.g., the minimum or product), then $\mathcal{D}(A_i, A_j) \leq \mathcal{E}(C_i, C_j)$ is satisfied in two situations:

- $\mathcal{E}(C_i, C_j) > 0$; then Supp $C_i$ = Supp $C_j$, which is rather unusual,
- $\mathcal{E}(C_i, C_j) = 0$; then $\mathcal{D}(A_i, A_j) = 0$, Supp $A_i \cap$ Supp $A_j = \emptyset$; for continuous degrees of membership, strong completeness is violated.



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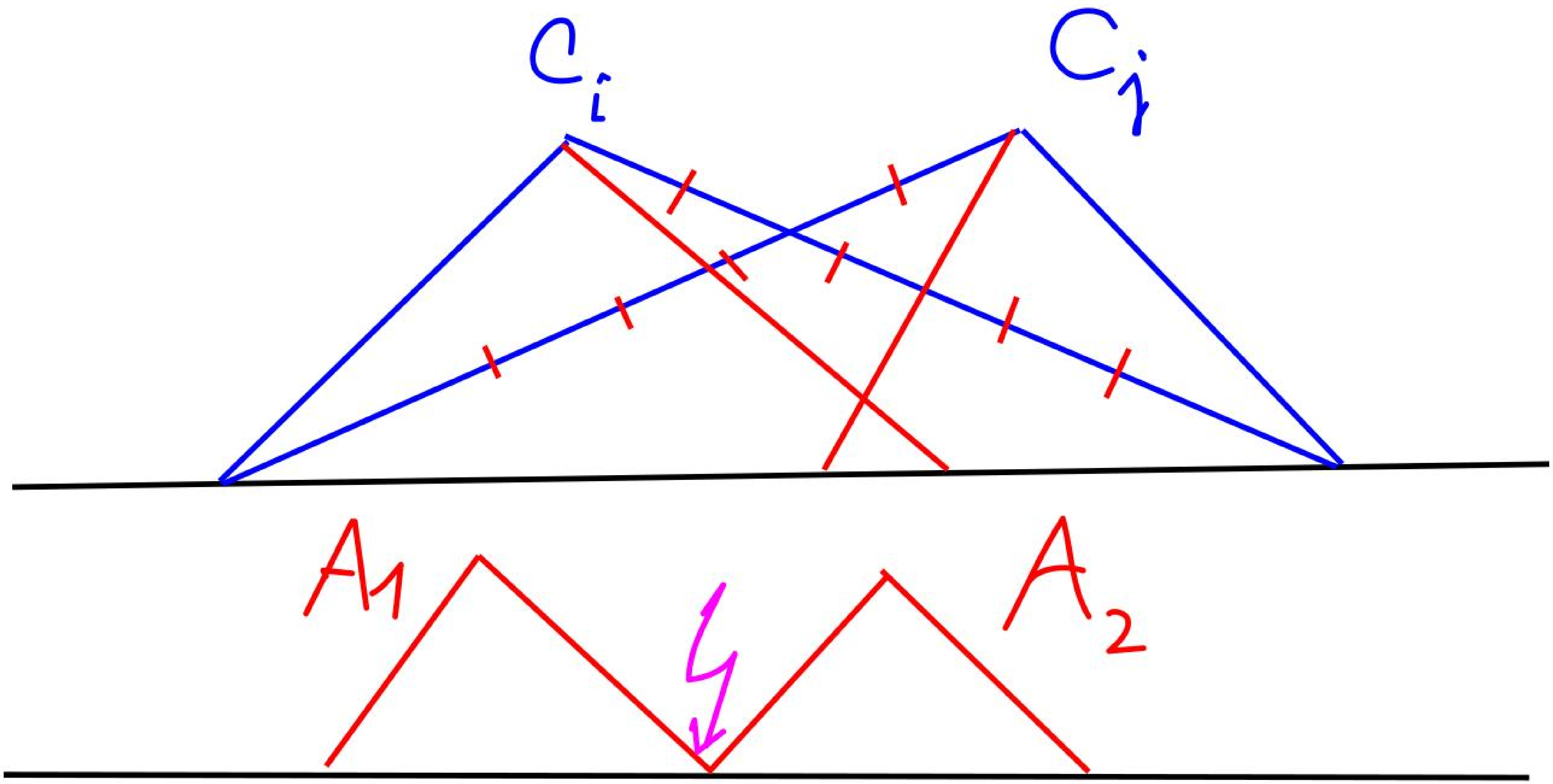

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This problem does not occur if $\underset{\cdot}{\wedge}$ has zero divisors (e.g., the Łukasiewicz t-norm).

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This problem does not occur if $\dot{\wedge}$ has zero divisors (e.g., the Łukasiewicz t-norm).

However, this choice may easily violate the strong completeness [Moser, Navara 1999].

# Correctness of residuum-based controller

**Theorem**: $\forall j : \Phi_{\mathsf{RES}}(A_j) \le C_j$ .

**Proof**: $X := A_j$ ,

$$\begin{aligned}
\Phi_{\mathsf{RES}}(A_j)(y) &= \sup_x \big(A_j(x) \underset{\cdot}{\wedge} \min_i (A_i(x) \underset{\cdot}{\rightarrow} C_i(y))\big) \\
&\le \sup_x \big(A_j(x) \underset{\cdot}{\wedge} (A_j(x) \underset{\cdot}{\rightarrow} C_j(y))\big) \le C_j(y)\,.
\end{aligned}$$

**Theorem**: If there is a fuzzy relation $R$ such that $\forall j : A_j \underset{\cdot}{\circ} R = C_j$ , then also $R_{\mathsf{RES}}$ satisfies these equalities (and it is the largest solution).

**Proof**: $\forall j \ \forall x \ \forall y :$

$$\begin{aligned}
A_j(x) \underset{\cdot}{\wedge} R(x,y) &\le C_j(y) \\
R(x,y) &\le A_j(x) \underset{\cdot}{\rightarrow} C_j(y) \\
R(x,y) &\le \min_i \big(A_i(x) \underset{\cdot}{\rightarrow} C_i(y)\big) = R_{\mathsf{RES}}(x,y)\,,
\end{aligned}$$

$$C_j = A_j \underset{\cdot}{\circ} R \le A_j \underset{\cdot}{\circ} R_{\mathsf{RES}} \le C_j\,.$$

What happens if correctness is violated?

Nothing serious, this is usually accepted and possibly compensated during the tuning.

However, it causes a distorted interpretation of (possibly good) control rules.

## Correctness of residuum-based controller



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Nothing serious, this is usually accepted and possibly compensated during the tuning.

However, it causes a distorted interpretation of (possibly good) control rules.

## An alternative: CFR (Controller with conditionally firing rules) [Moser, Navara 2002]



**1st generalization of Mamdani–Assilian controller**:
$\varrho\colon [0,1] \to [0,1]$ ... increasing bijection, e.g., $\varrho(t) = t^r$, $r > 1$, or piecewise linear.
Transformation of membership degrees in the input space $\mathcal{X}$.

The degrees of overlapping, $\mathcal{D}(A_i \circ \varrho, A_j \circ \varrho)$, may be made arbitrarily small.

**2nd generalization of Mamdani–Assilian controller**:
$\sigma\colon [0,1] \to [c,1]$ ... increasing bijection
$(0 < c < 1)$.
Transformation of membership degrees in the output space $\mathcal{Y}$.
Output $Y \circ \sigma$ has to be transformed back by $\sigma^{[-1]}$,
so the inference rule is not compositional
(however, the computational complexity remains of the same order).

The degrees of equality, $\mathcal{E}(C_i \circ \sigma, C_j \circ \sigma)$, may be made arbitrarily large.

We may satisfy $\mathcal{D}(A_i \circ \varrho, A_j \circ \varrho) \le \mathcal{E}(C_i \circ \sigma, C_j \circ \sigma)$.

**Problem 1**: $\mathcal{D}(X \circ \varrho, A_i \circ \varrho)$ becomes also small, causing "irrelevant outputs" and violating strong completeness.

**Problem 2**: Correctness and strong completeness are "almost contradictory" for the Mamdani–Assilian controller; sometimes they cannot be satisfied simultaneously for any compositional inference rule.

## An alternative: CFR (Controller with conditionally firing rules)

So far, we obtained a special case of the generalized FATI inference rule, where $\quad \pi_i(a,b) = \varrho(a) \mathbin{\dot\wedge} \sigma(b)\,, \quad \beta = \max\,, \quad \kappa(a,b) = \varrho(a) \mathbin{\dot\wedge} b\,, \quad Q = \sup \circ \sigma^{[-1]}\,.$

However, we need:

**3rd generalization of Mamdani–Assilian controller**:
For the **degree of firing** in the inference rule, replace the **degree of overlapping** $\mathcal{D}(X, A_i)$ with the normalized value — **degree of conditional firing**

$$\mathcal{C}_i(X) = \frac{\mathcal{D}(X, A_i)}{\max\limits_j \mathcal{D}(X, A_j)}\,.$$

All the above requirements (in particular correctness and crisp correctness) are satisfied if [Moser, Navara 2002, Navara, Šťastný 2002]:

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[C2] Each point of the input space belongs to the support of some antecedent.

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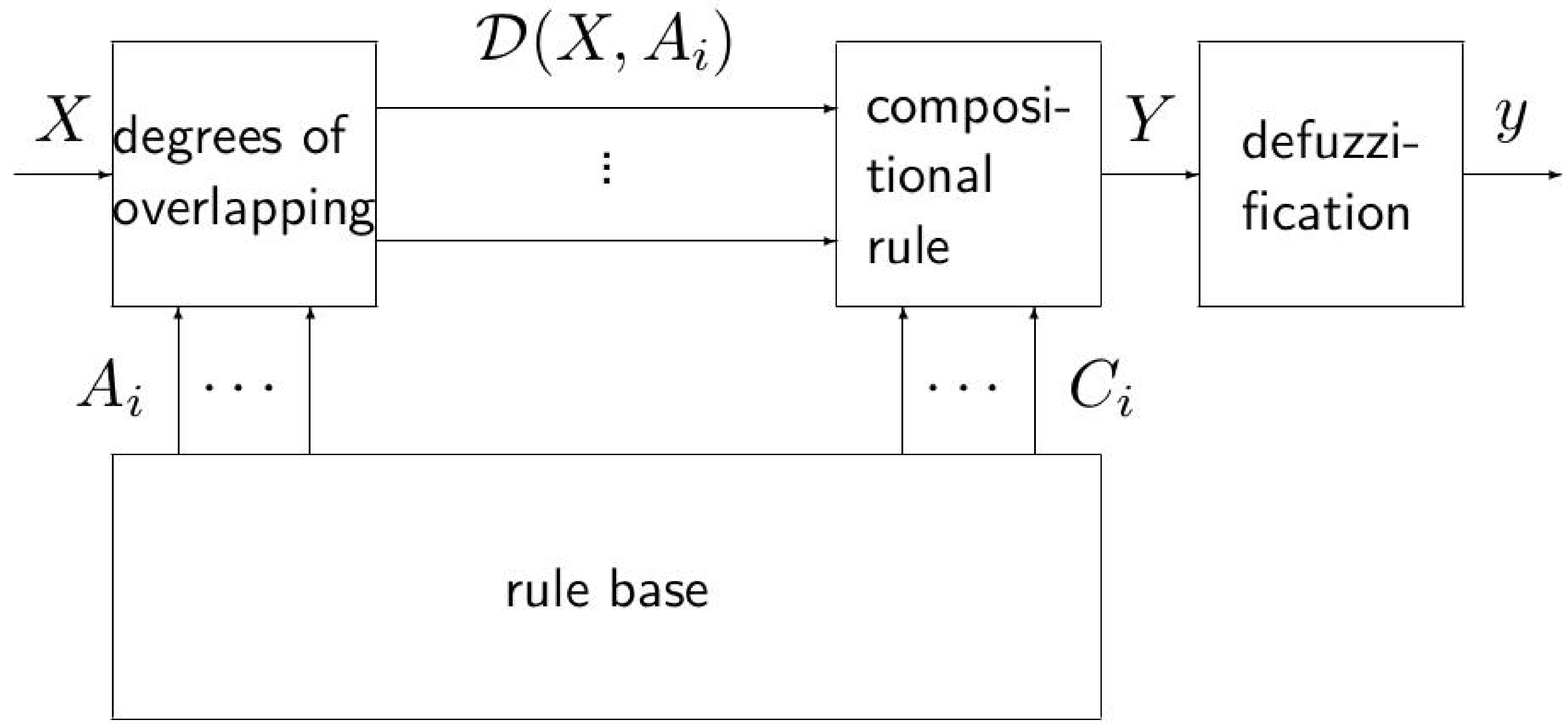
[C1] Each antecedent is normal.

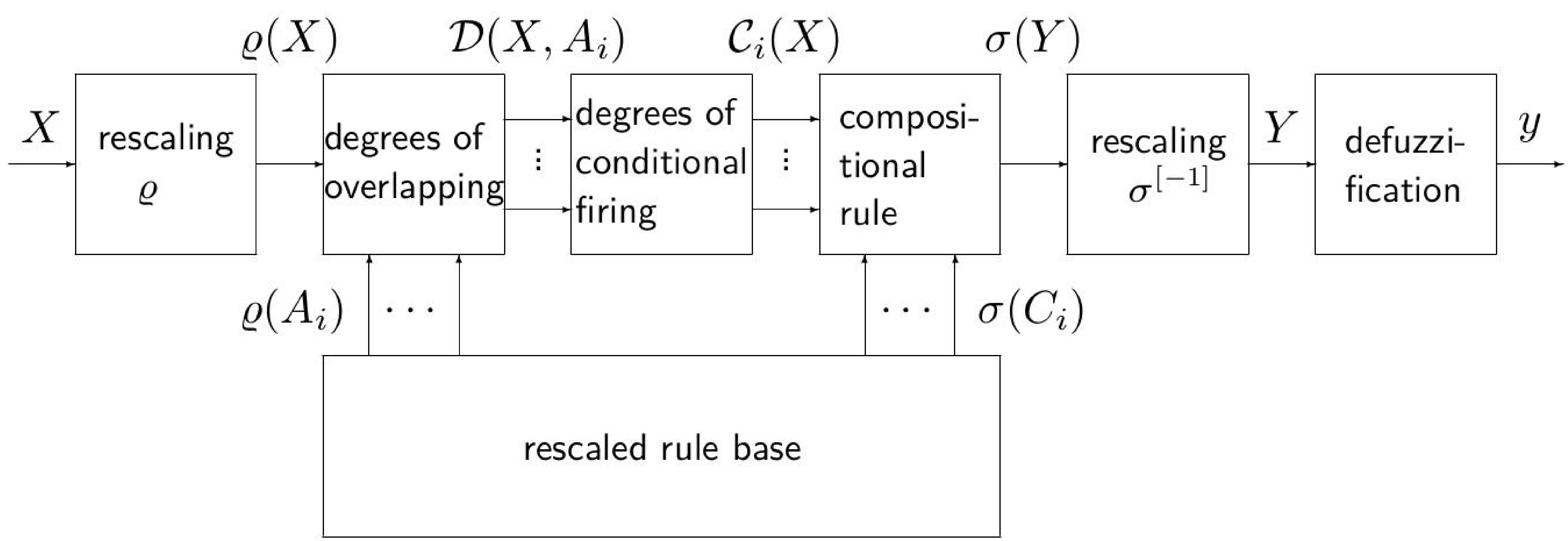
[C2] Each point of the input space belongs to the support of some antecedent.

[C3] No consequent is covered by the maximum all other consequents.

[C4] “Weak disjointness of antecedents": $\exists c < 1 : A_i(x) \dot{\wedge} A_j(x) < c$ whenever $i \neq j$.

# Comparison of Mamdani–Assilian and CFR controller — block diagrams



# Sample problem: ball on beam (ball on plate)

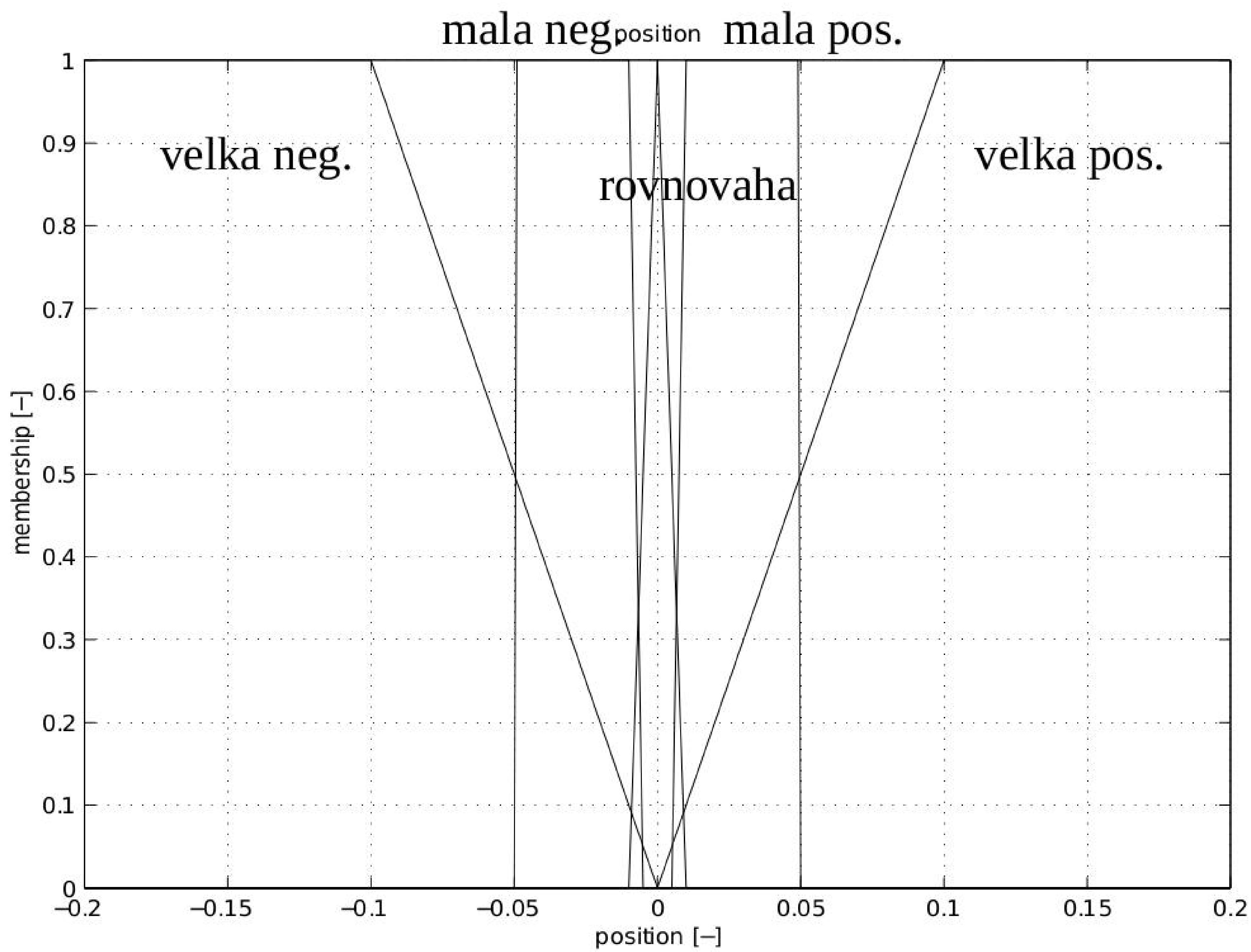
We want to stabilize a position of a ball by leaning a plate on which it lies.

Static friction is considered ($\Rightarrow$ non-linearity).

Solution due to [Šťastný 2001].

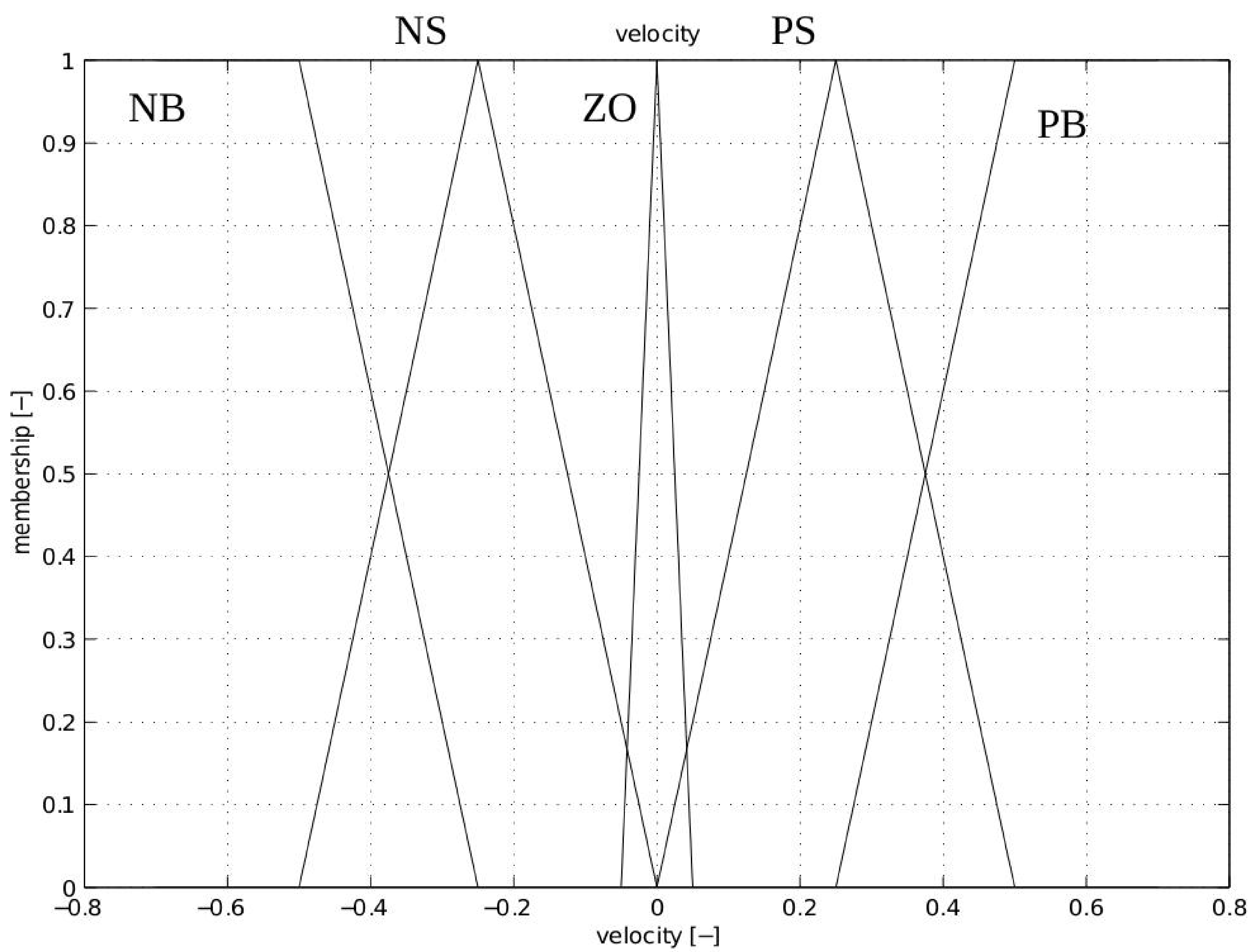
# Example: Comparison of Mamdani–Assilian and CFR controller — position (premises)





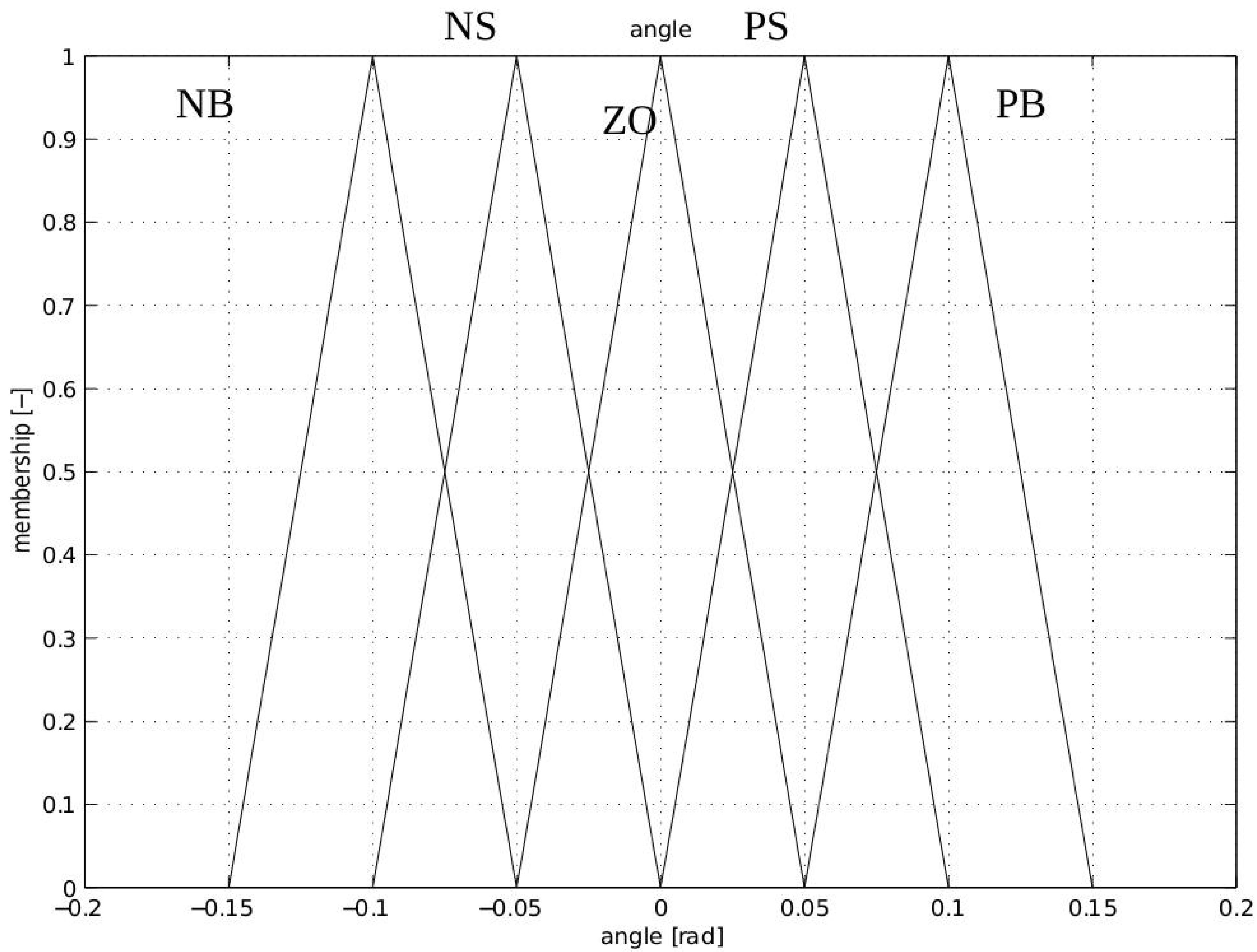
# Example: Comparison of Mamdani–Assilian and CFR controller — velocity (premises)





# Example: Comparison of Mamdani–Assilian and CFR controller — angle (consequents)





## Example: Comparison of Mamdani–Assilian and CFR controller — rules

Angle:

| position / velocity | NB | NS | ZO | PS | PB |
|---|---|---|---|---|---|
| NB | PB | PB | PB | PB | PS |
| NS | PB | PS | PS | PS | ZO |
| ZO | PB | PB | ZO | NB | NB |
| PS | ZO | NS | NS | NS | NB |
| PB | NS | NB | NB | NB | NB |

# Example: Comparison of Mamdani–Assilian and CFR controller — quality of control



| criterion | Mam. controller | CFR controller |
|---|---|---|
| maximum overshoot [m] $\sigma$ | - | - |
| asymptotic value [m] $y_\infty$ | -0.0021 | 0.0012 |
| number of extremes [-] | - | - |
| transient time [s] | 3.56 | 3.05 |
| cumulative quadratic error [ms] | 0.0552 | 0.0569 |

Ball on plate, initial position +0.25, simulation time $5\,s$ — till steady state. Smaller values – better control. $T_{OUT} = 100\,ms$.

| criterion | Mam. controller | CFR controller |
|---|---|---|
| maximum overshoot [m] $\sigma$ | - | - |
| asymptotic value [m] $y_\infty$ | -0.0052 | -0.0006 |
| number of extremes [-] | - | - |
| transient time [s] | 18.06 | 17.22 |
| cumulative quadratic error [ms] | 23.12 | 22.34 |

Ball on plate, initial position +2.00, simulation time $20\,s$ — till steady state. Smaller values – better control. $T_{OUT} = 100\,ms$.

# Example: Comparison of Mamdani–Assilian and CFR controller — quality of control



| criterion | Mam. controller | CFR controller |
|---|---|---|
| maximum overshoot [m] $\sigma$ | 0.35 | 0.35 |
| asymptotic value [m] $y_\infty$ | -0.0032 | -0.0041 |
| number of extremes [-] | 1 | 1 |
| transient time [s] | 13.49 | 11.39 |
| cumulative quadratic error [ms] | 0.523 | 0.474 |

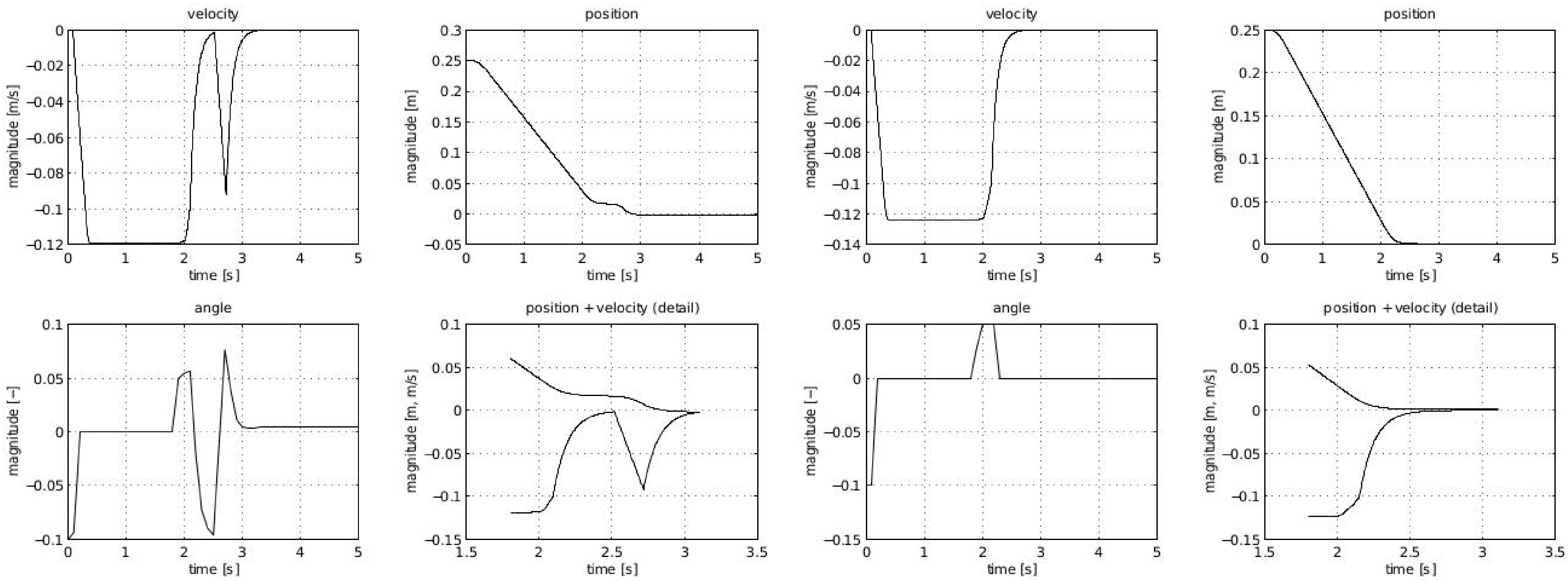
Ball on plate, initial speed $0.5ms^{-1}$, simulation time $15\,s$ — till steady state. Smaller values – better control. $T_{OUT} = 50\,ms$.

| criterion | Mam. controller | CFR controller |
|---|---|---|
| maximum overshoot [m] $\sigma$ | 0.346 | 0.346 |
| asymptotic value [m] $y_\infty$ | 0.0051 | 0.0034 |
| number of extremes [-] | 1 | 1 |
| transient time [s] | 14.8 | 11.1 |
| cumulative quadratic error [ms] | 0.583 | 0.441 |

Ball on plate, initial speed $0.5ms^{-1}$, simulation time $15\,s$ — till steady state. Smaller values – better control. $T_{OUT} = 5\,ms$.

# Example: Comparison of Mamdani–Assilian and CFR controller — outputs






Typical outputs of Mamdani–Assilian controller (left) and CFR controller (right).

# Problems of implementation of CFR controller



**Software implementation:** only three new blocks requiring a few lines of source code. The computational complexity slightly increases, but its order remains unchanged.

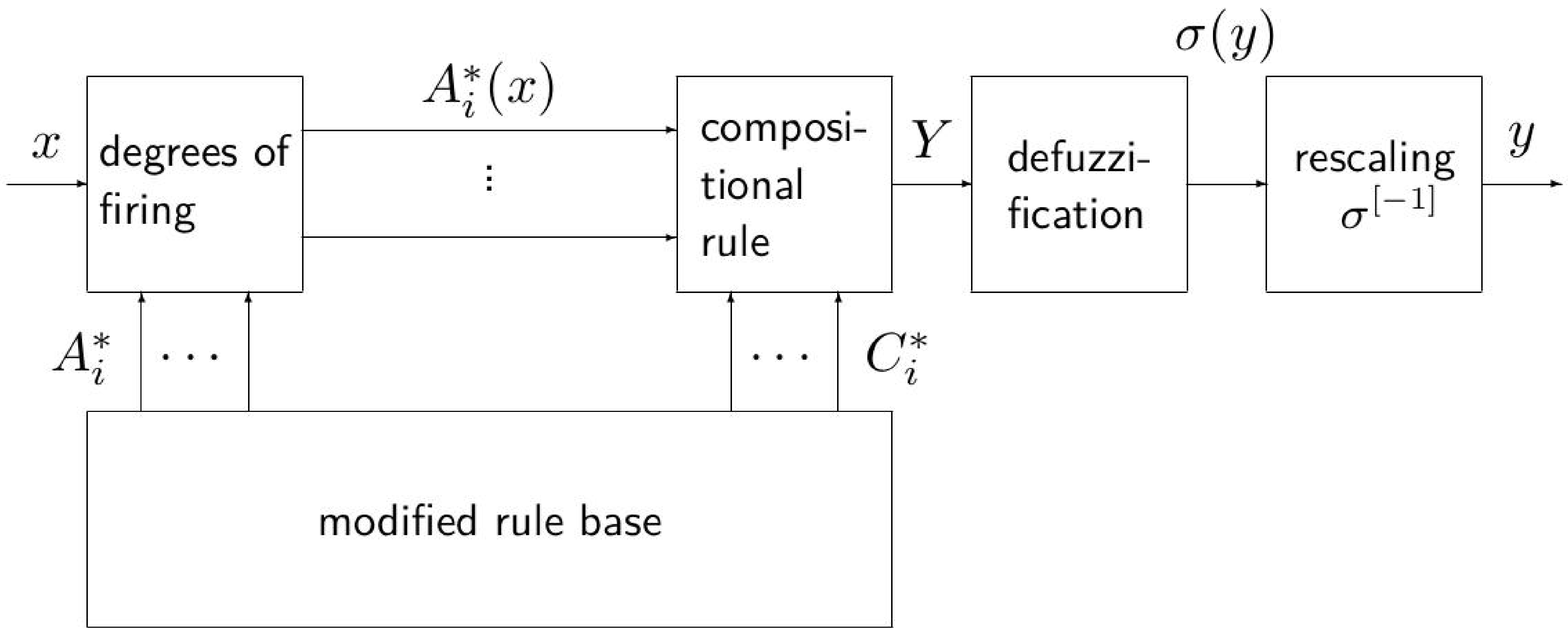
**Hardware implementation:** Requires to add an additional block inside the current structure, thus a totally new design of an integrated circuit - **expensive!**

Looking for a possibility to achieve the same control action using current fuzzy hardware and a **modified rule base**, we have found [Amato, Di Nola, Navara 2003]:

1. it is not possible to substitute the CFR controller in its full generality, but
2. this is possible for crisp input variables.

This case is still of much importance, because it covers most of applications; in fact, current fuzzy hardware works only with crisp inputs.

# Hardware implementation of CFR controller



# Conclusion

- We formulated well motivated axioms for fuzzy controllers (approximators). They cannot be satisfied by any controller using the classical compositional rule of inference (including the Mamdani–Assilian controller). Our generalized controller satisfies them under very general conditions.
- Practical experiments show that our controller allows to achieve better results with the same rule database.
- The computational efficiency is basically the same as that of the Mamdani–Assilian controller.
- New results allow to transform the rule base (automatically) so that the current fuzzy hardware could be used to implementation of our controller, although its performance could not be achieved by the original Mamdani–Assilian controller.

## Initial rule base

Can be obtained by

- asking an expert,

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- a template for a similar problem.