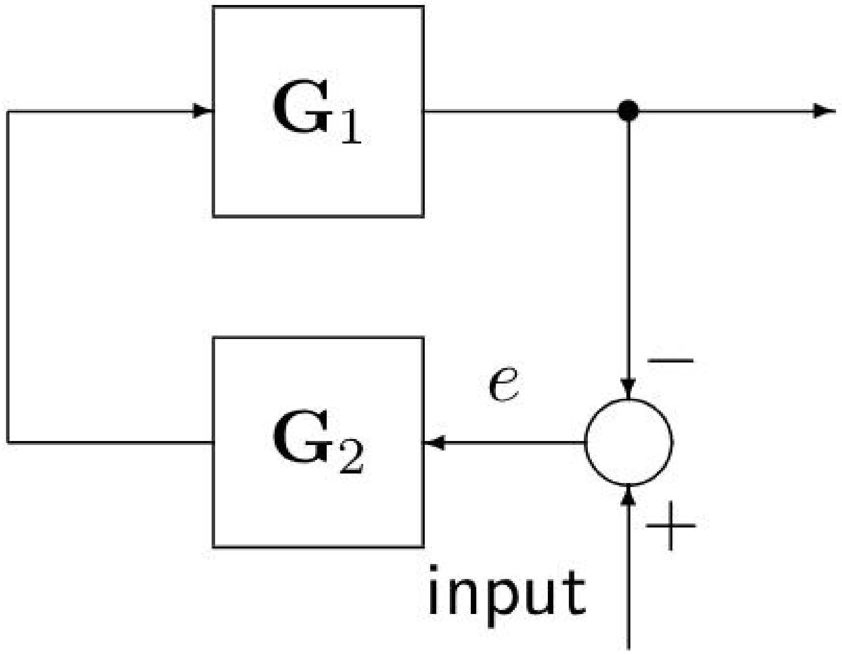
Classical controller design

We usually put the controller in the feedback loop



$$\mathbf{G}(s) = \left(\mathbf{I} \cancel{+} \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1} \mathbf{G}_2(s)\,\mathbf{G}_1(s)$$

The stability of the whole loop is influenced by the factor $\left(\mathbf{I} \cancel{+} \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}$
where $\mathbf{G}_1(s)$ (the controlled system) is given
and $\mathbf{G}_2(s)$ (the controller) can be chosen almost arbitrarily

Using the above analysis, we can decide the stability of the loop with the proposed controller (difficult)

The task is easier if we have more information than the output of the controlled system, in particular if we can measure the **states**; then a feedback from states allows – in its extreme (theoretical) form – to achieve arbitrary dynamics of the control loop

Generally, the more information we have the better the control behaviour can be

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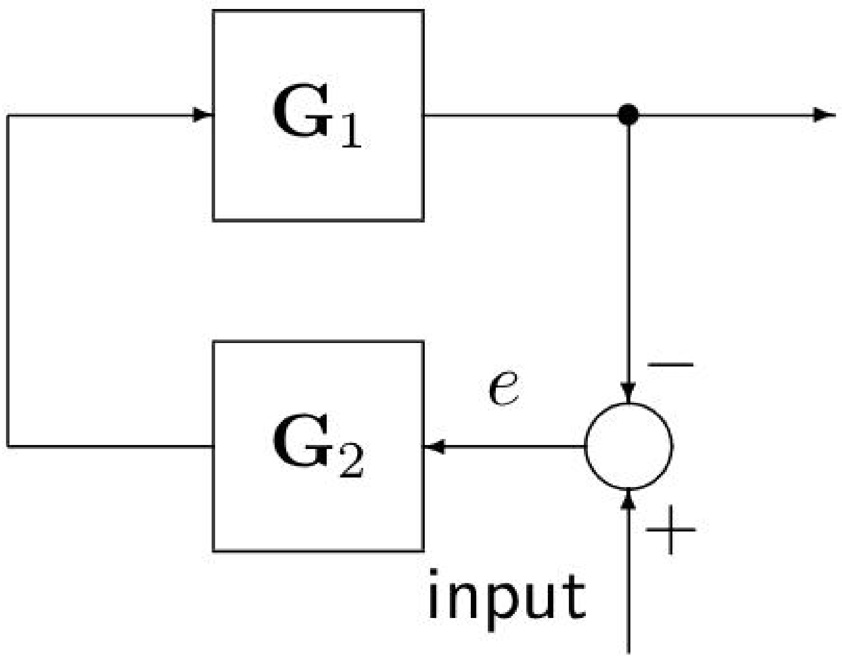

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1 Stabilizing an inverted pendulum (cartpole problem)

Using acceleration a, we want to stabilize an inverted pendulum of efficient length ℓ. The angle φ satisfies the ODE

$$\ell\,\varphi'' - g\,\varphi + a = 0$$

where g is the acceleration of gravity, with initial conditions

$\varphi(0) = \varphi_0, \;\; \varphi'(0) = \omega_0.$

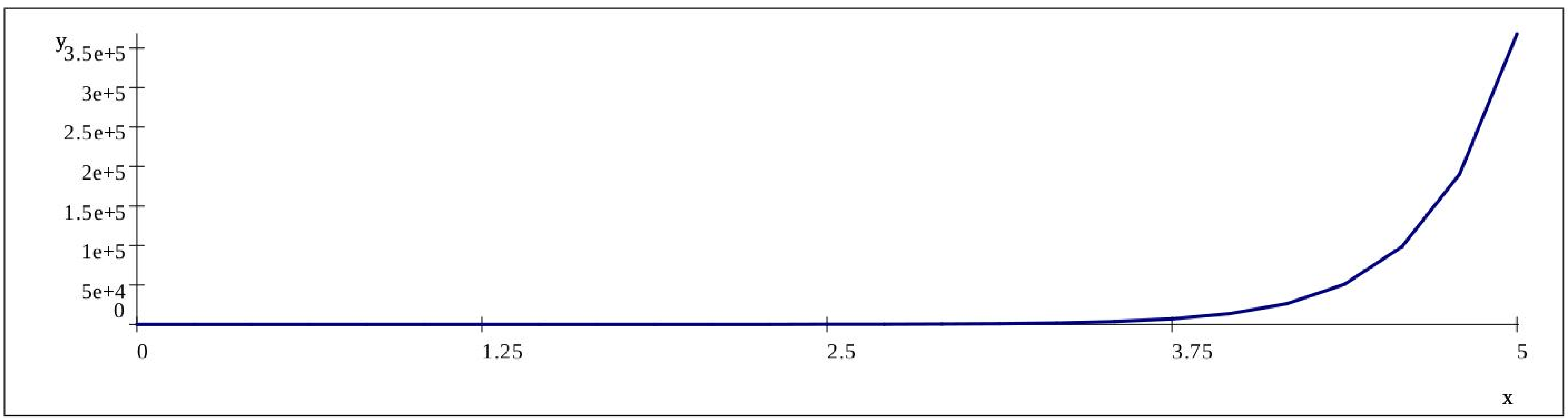
In examples, we use $g = 10, \;\; \ell = 1, \;\; \varphi_0 = 0.1, \;\; \omega_0 = 0.$

1.1 No controller $\quad a = 0: \quad \ell\,\varphi'' - g\,\varphi = 0$

The system

$$\begin{aligned} \varphi'' - 10\,\varphi &= 0 \\ \varphi(0) &= 0.1 \\ \varphi'(0) &= 0 \end{aligned}$$

is unstable, the exact solution is: $0.05e^{t\sqrt{10}} + 0.05e^{-t\sqrt{10}}$

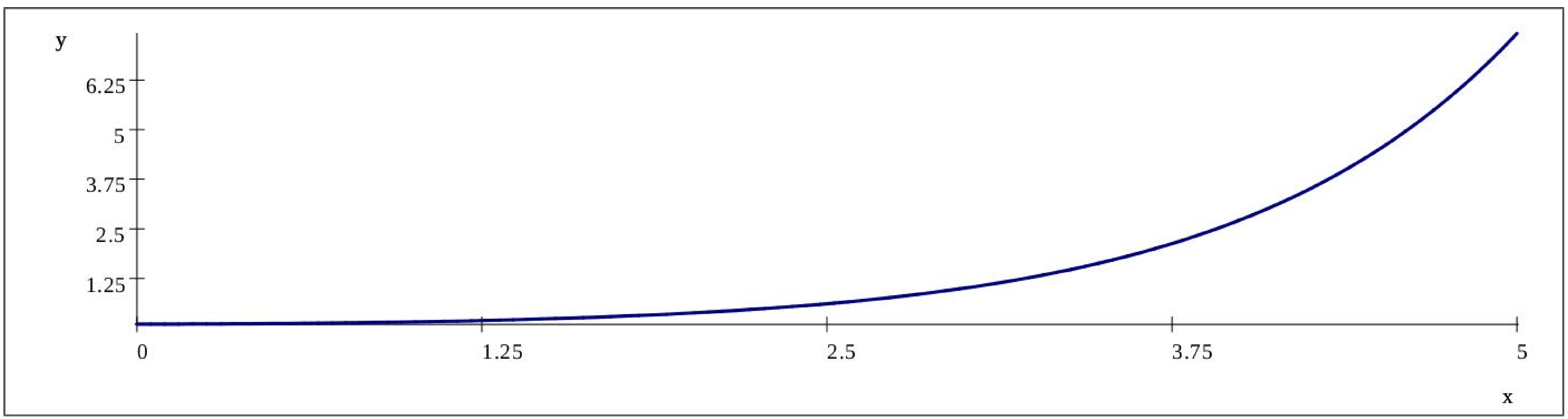


1.2 P controller $a = p\,\varphi$: $\ell\,\varphi'' - g\,\varphi + p\,\varphi = 0$

The system is unstable for all p, e.g.

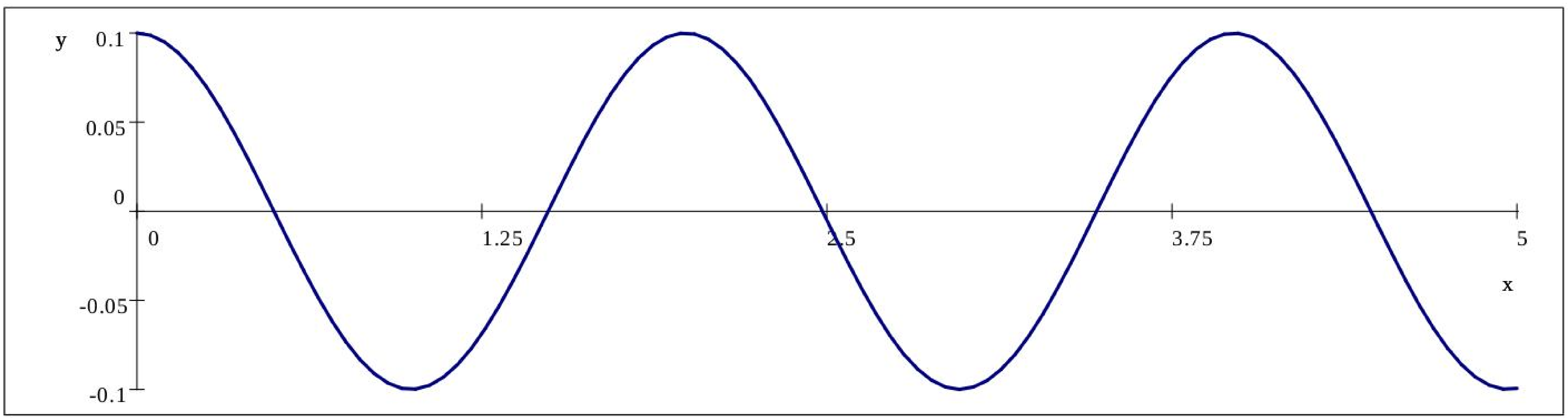
$$\varphi'' - 10\,\varphi + 9\,\varphi = 0$$
$$\varphi(0) = 0.1$$
$$\varphi'(0) = 0$$

the exact solution is: $0.05e^{t} + 0.05e^{-t}$



$$
\begin{aligned}
\varphi'' - 10\,\varphi + 20\,\varphi &= 0 \\
\varphi(0) &= 0.1 \\
\varphi'(0) &= 0
\end{aligned}
$$

the exact solution is: $0.1\cos t\sqrt{10}$



1.3 PI controller $a = p\,\varphi + i\,\int_0^t \varphi(\tau)\,d\tau$

$$\ell\,\varphi'' - g\,\varphi + p\,\varphi + i\,\int_0^t \varphi(\tau)\,d\tau = 0$$

Difficult to solve, the order increased.

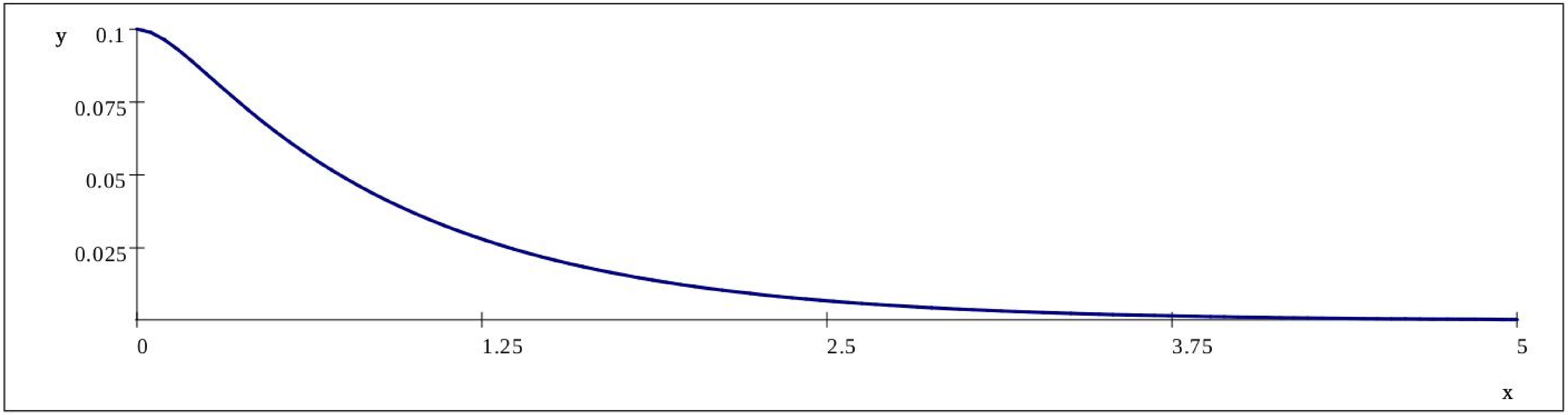
1.4 PD controller $a = p\,\varphi + d\,\varphi'$

$$\ell\,\varphi'' - g\,\varphi + p\,\varphi + d\,\varphi' = 0$$

It can be stable, e.g.

$$\begin{aligned}\varphi'' - 10\,\varphi + 20\,\varphi + 10\,\varphi' &= 0\\ \varphi(0) &= 0.1\\ \varphi'(0) &= 0\end{aligned}$$

the exact solution is: $0.114\,55e^{t\left(\sqrt{15}-5\right)}-1.455\,0\times 10^{-2}e^{t\left(-\sqrt{15}-5\right)}$



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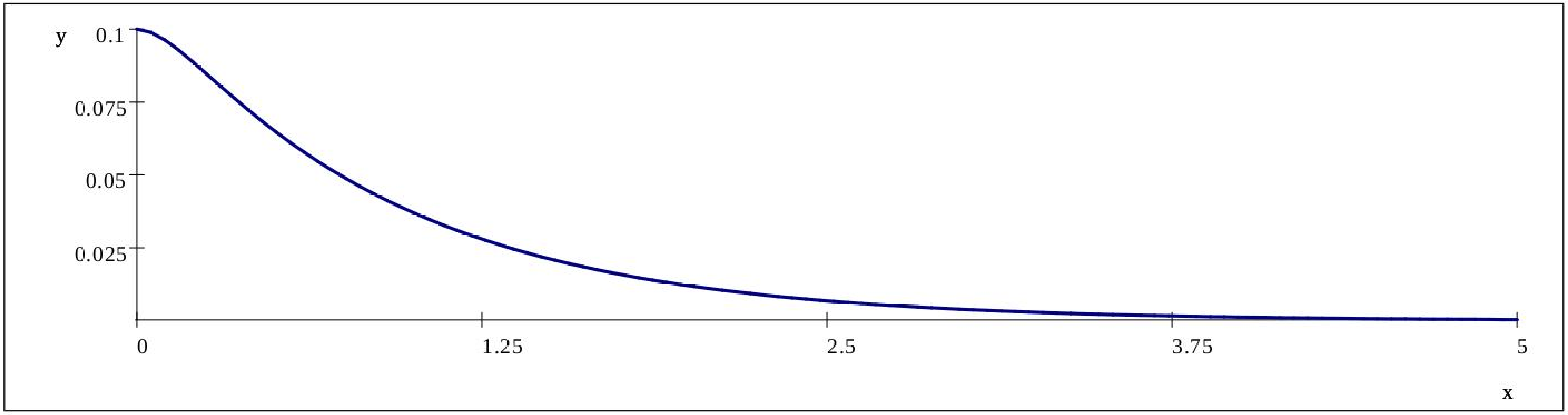
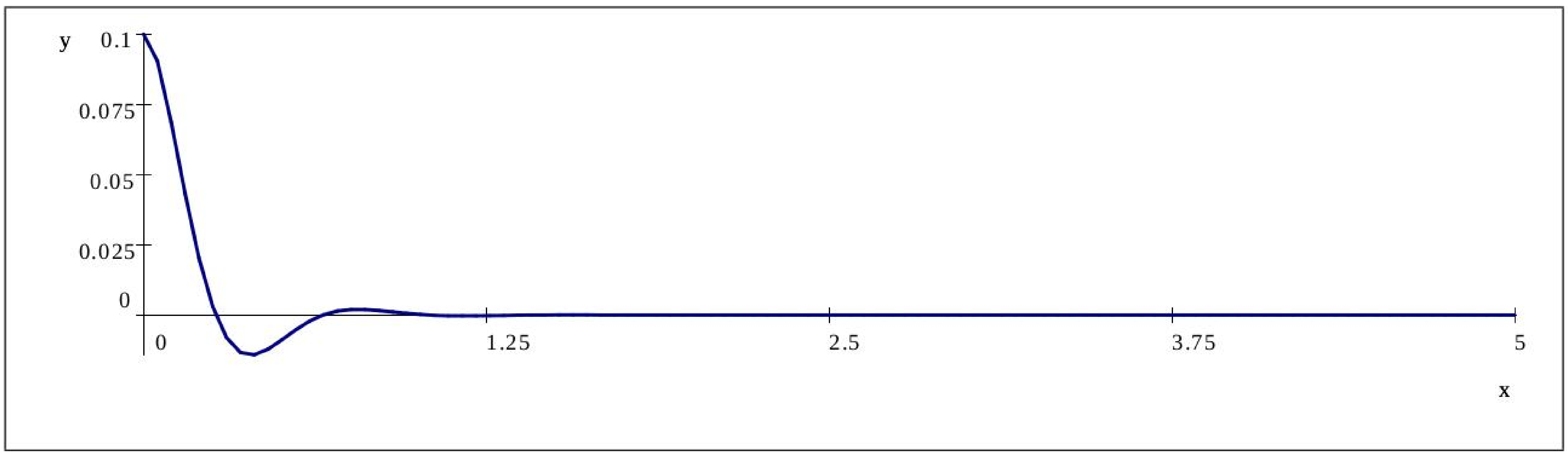
$$\begin{aligned} \varphi'' - 10\,\varphi + 20\,\varphi + 10\,\varphi' &= 0 \\ \varphi(0) &= 0.1 \\ \varphi'(0) &= 0 \end{aligned}$$

the exact solution is: $0.11455e^{t\left(\sqrt{15}-5\right)} - 1.4550\times 10^{-2}e^{t\left(-\sqrt{15}-5\right)}$



$$\varphi'' - 10\,\varphi + 100\,\varphi + 10\,\varphi' = 0$$
$$\varphi(0) = 0.1$$
$$\varphi'(0) = 0$$

the exact solution is: $0.1e^{-5t}\cos t\sqrt{65} + 6.2017 \times 10^{-2}e^{-5t}\sin t\sqrt{65}$



$$\varphi'' - 10\,\varphi + 35\,\varphi + 10\,\varphi' = 0$$
$$\varphi(0) = 0.1$$
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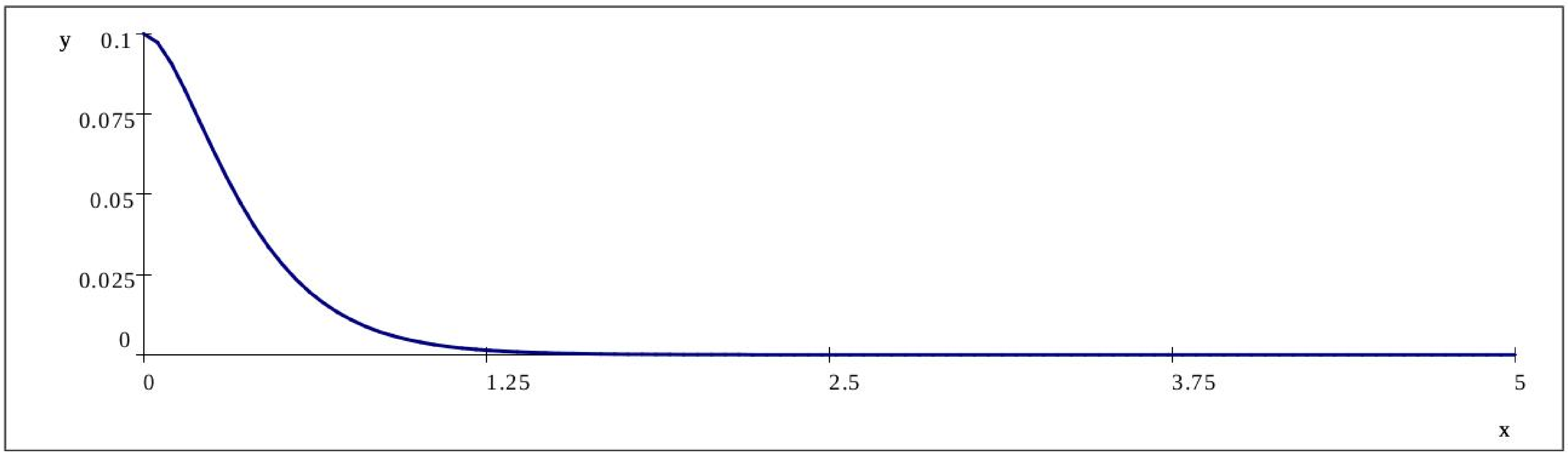

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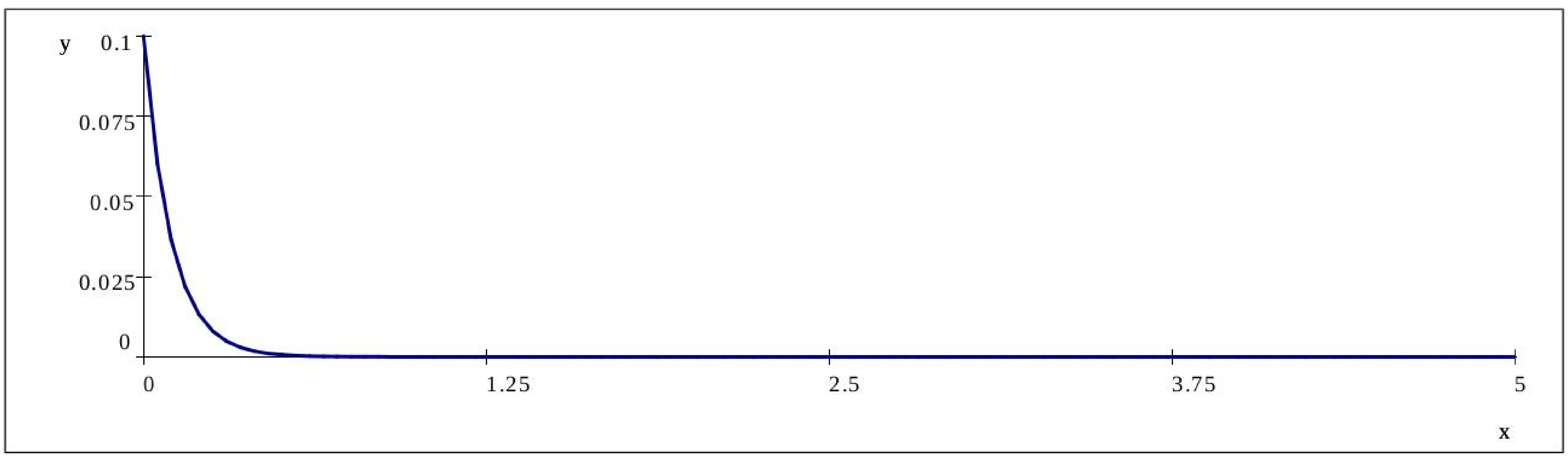

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- Parameters are not precisely known (or it is difficult to measure them)
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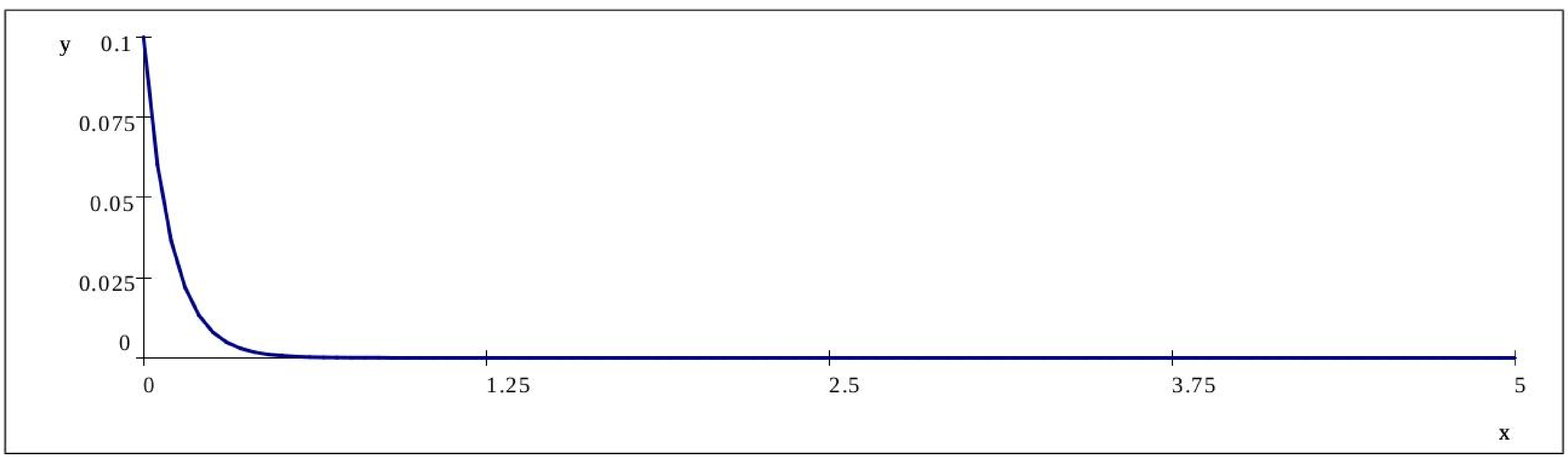


2 Why these strange results?

We have to look at the characteristic values of the systems, these are roots of the denominator of the transfer function of the feedback loop,

$$G(s) = \frac{G_1(s)}{1 - G_2(s)\, G_1(s)}$$

where

$$G_1(s) = \frac{1}{g - \ell\, s^2}$$

and $G_2(s)$ is the transfer function of the controller, hence

$$G(s) = \frac{-1}{\ell\, s^2 - g + G_2(s)}$$

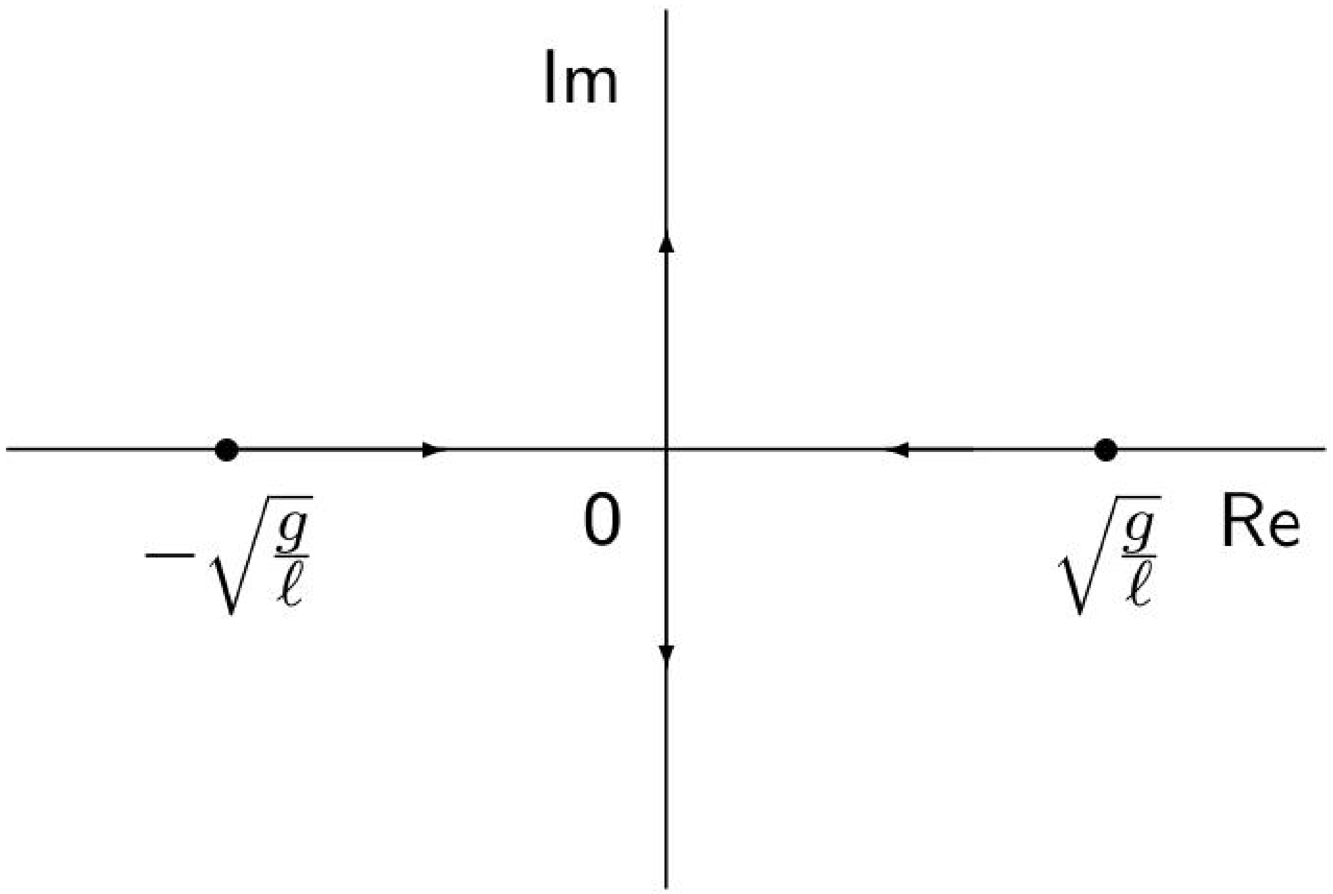
and we study the roots of $\ell\, s^2 - g + G_2(s)$

2.1 No controller $\quad G(s) = 0$: $\quad \ell\, s^2 - g = 0$

$s = \pm\sqrt{\frac{g}{l}}$ (in the real domain)

2.2 P controller $\quad G(s) = p$: $\quad \ell\, s^2 - g + p = 0$

$s = \sqrt{\frac{g - p}{l}}$ (two-valued square root in the complex domain)



The roots are symmetric w.r.t. 0, so stability cannot be achieved

2.3 PI controller $\quad G(s) = p + \dfrac{i}{s}$

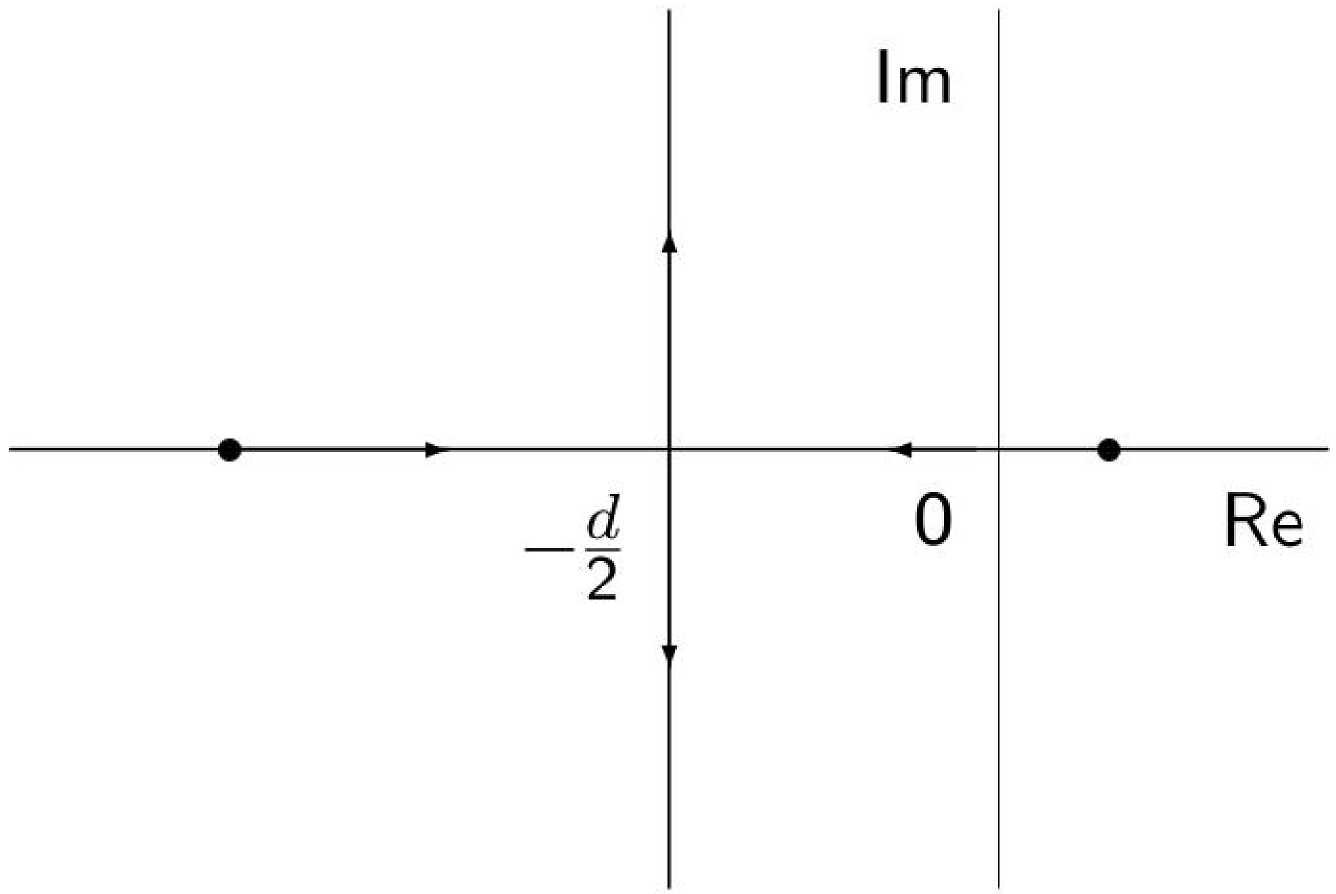
$$\ell\, s^2 - g + p + \frac{i}{s} = 0$$

Difficult to solve (3rd order, even for a drastically simplified task)

2.4 PD controller $\quad G(s) = p + d\, s$

$$\ell\, s^2 - g + p + d\, s \; = 0$$

$$s = \frac{-d + \sqrt{d^2 - 4\, \ell\, (p - g)}}{2\, \ell}$$



The roots are symmetric w.r.t. $\dfrac{-d}{2\,\ell}$, so stability can be achieved

2.5 State-feedback controller $\quad G(s) = p + d\,s + c\,s^2$

$$\ell\,s^2 - g + p + d\,s + c\,s^2 = 0$$

Any form is theoretically possible, e.g., $c = -\ell,\ p = 20,\ d = 1$:

$$\begin{aligned} s^2 - 10 + 20 + s - s^2 &= 0 \\ s &= -10 \end{aligned}$$

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Brief history of fuzzy control

[Zadeh 1973] suggested the use of fuzzy logic in control (he already contributed to the development of the classical control theory)

[Mamdani, Assilian 1975]: the first fuzzy controller (of a steam engine)

[Holmblad 1982]: a fuzzy controller of a cement kiln (high non-linearity, many variables, manual control used before)

[Sugeno 1985]: prototypes of other industrial applications

[Yasunubo et al. 1983]: a fuzzy controller of the Sendai Underground (operating since 1987)

A boom of fuzzy controllers in 80's and 90's, mainly in Japan (now mainly washing machines, vacuum cleaners, camcorders, etc.)

Now it is time to test whether fuzzy control may infiltrate in more difficult and demanding applications

Basic ideas and notions of fuzzy control

Inputs of a fuzzy controller:

- desired output values
- actual output values
- possibly internal states of the controlled system
- event. additional information from the user, usually linguistic

Input variables are coordinates in the **input space**, $\mathcal{X}$, usually a convex subset of R^μ

Outputs of a fuzzy controller:

- control actions (inputs of the controlled system)
- event. additional information for the user

Output variables are coordinates in the **output space**, $\mathcal{Y}$, usually a convex subset of R^ν

Crisp controller

A classical controller performs a mapping $f\colon \mathcal{X} \to \mathcal{Y}$
It can be represented by a crisp subset of $\mathcal{X} \times \mathcal{Y}$, namely

$$\{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid y = f(x)\}$$

and by a (crisp) membership function $R\colon \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$

$$R(x, y) = \begin{cases} 1 & \text{if } y = f(x), \\ 0 & \text{otherwise} \end{cases}$$

Motivation of a fuzzy controller

Sometimes a human expert (or other source, e.g., data mining) can give us hints in the form

if input is ... **then** output is ... **and**

...

if $x \in A_n$ **then** $y \in C_n$

(rule base of **if-then rules**)

The rules are vague, often with unsharp boundaries of applicability

Fuzzy controller

Zadeh's suggestion (1973): Express the information from the rule base using fuzzy sets, as a fuzzy relation $R\colon \mathcal{X} \times \mathcal{Y} \to [0,1]$
(a fuzzy subset of $\mathcal{X} \times \mathcal{Y}$, $R \in \mathcal{F}(\mathcal{X} \times \mathcal{Y})$)
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Moreover, the internal **inference mechanism** can work with fuzzy subsets of the input/output space (instead of points) and map fuzzy subsets of the input space $\mathcal{X}$ onto fuzzy subsets of the output space $\mathcal{Y}$,

$$\Phi: \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y})$$

The input can be fuzzy, but it is often crisp; sometimes a crisp input is fuzzified

The output can be fuzzy, but usually a crisp value is required; a **defuzzification** $\Delta: \mathcal{F}(\mathcal{Y}) \to \mathcal{Y}$ takes place as the final step

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For fuzzy inference itself, we need a correspondence between **fuzzy** subsets of the input and output spaces

Basic notions of a fuzzy controller

Rule database:

if X is A_1 **then** Y is C_1 **and**

...

if X is A_n **then** Y is C_n

where

$X \in \mathcal{F}(\mathcal{X})$ is a fuzzy input

$Y = \Phi(X) \in \mathcal{F}(\mathcal{Y})$ is the corresponding fuzzy output

$A_i \in \mathcal{F}(\mathcal{X})$, $i = 1, \ldots, n$, are **antecedents** (**premises**) which can be interpreted as

- assumptions,
- domains of applicability, or
- typical fuzzy inputs

$C_i \in \mathcal{F}(\mathcal{Y})$, $i = 1, \ldots, n$, are **consequents** (**conclusions**) expressing the desired outputs

Dimensionality

Antecedents are subsets of multi-dimensional spaces

They carry information about several variables (and so do the consequents)

Usually they are decomposed to conjunctions (cylindric extensions) of one-dimensional fuzzy sets

Then the rules attain the form

if A_1 is A_{i1}
  and ...
  and A_μ is $A_{i\mu}$
then C_1 is C_{i1}
  and ...
  and C_ν is $C_{i\nu}$

$i = 1, \ldots, n$

If an antecedent has a more complex shape (non-convex), we may cover it approximately by several rules of the above form

Simplifying assumptions

1. We ignore the conjunctions (cylindric extensions) and admit arbitrary shapes of antecedents

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2. We decompose the output to single variables considered independently. Without loss of generality, we restrict attention to MISO (Mulitple Input Single Output) systems

Compositional rule of inference

The rule base is represented by a fuzzy relation $R \in \mathcal{F}(\mathcal{X} \times \mathcal{Y})$
The output, Y, is obtained by a composition of R with the input, X:

$$Y = \Phi(X) = X \underset{\cdot}{\circ} R$$

$$Y(y) = \sup_{x \in \mathcal{X}} \big(R(x, y) \underset{\cdot}{\wedge} X(x) \big)$$

where $\underset{\cdot}{\wedge}$ is a t-norm (fuzzy conjunction); different choices are possible, but we shall restrict to **continuous t-norms**

The supremum is the standard t-conorm; it should not be replaced by another t-conorm (because it may have uncountably many arguments)

How to derive the fuzzy relation R from the rule base

The most natural idea: **Residuum-based fuzzy controller**:

$$R_{\mathsf{RES}}(x,y) = \min_i \big(A_i(x) \underset{\cdot}{\to} C_i(y)\big)$$

where $\underset{\cdot}{\to}$ is a **fuzzy implication**, usually the **residuum** (**R-implication**) of $\underset{\cdot}{\wedge}$,

$$\alpha \underset{\cdot}{\to} \beta = \sup\{\gamma \in [0,1] \mid \gamma \underset{\cdot}{\wedge} \alpha \leq \beta\}$$

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