When Do We Need Quantum Probability Instead of the Classical One
(Kdy potřebujeme kvantovou pravděpodobnost místo klasické)

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Rozsah	**Volby**	Factum 1000	Median 1000	STEM 1000	CVVM 1000	Sanep 7000	Médea 1000	Aisa 1000
ČSSD	**20.5**	**22.8**	25.5	25.9	26	23.8	**22.2**	**23**
ANO	**18.7**	12.1	13	16.1	**16.5**	11.6	**16.9**	16
KSČM	**14.9**	**17.1**	**16**	**13.3**	18	16.9	11.8	**14**
TOP 09	**12**	**13.2**	**13**	**11.5**	9	**11.9**	9.6	**10.5**
ODS	**7.7**	**7.2**	**8**	**8.6**	**6.5**	**7.5**	5.5	**7**
Úsvit	**6.9**	3.7	4	**5.9**	5	5.3	**8.2**	**6**
KDU	**6.8**	**5.9**	**6**	4.5	5	5.7	**6.2**	**6**
SZ	**3.2**	**4.3**	**3**	**2.6**	2	**3.1**	**2.9**	**3**
SPOZ	**1.5**	4.7	5	2.6	3.5	5.2	3.7	4
ostatní	**7.8**	**9**	**6.5**	**9**	**8.5**	**9**	13	10.5
Účast	**59.5**	54	**60**	67	63	**59.3**	71	78
χ^2-test		≈ 0	≈ 0	$5 \cdot 10^{-6}$	$2 \cdot 10^{-12}$	≈ 0	≈ 0	$3 \cdot 10^{-10}$

Tučně jsou vyznačeny předpovědi, které se vejdou do 95% symetrického intervalového odhadu. χ^2**-test dobré shody** ukazuje, že u nejúspěšnější agentury lze hypotézu o shodě rozdělení zamítnout na hladině významnosti $5 \cdot 10^{-6}$, tj. takový výsledek bychom při shodě rozdělení dostali s pravděpodobností $1/200\,000$, u ostatních agentur ještě menší.

σ-algebra of sets, $\mathcal{L} \subseteq 2^U$:

- $U \in \mathcal{L}$
- $A \in \mathcal{L} \Longrightarrow U \setminus A \in \mathcal{L}$
- $\{A_i \mid i \in \mathbb{N}\} \subseteq \mathcal{L} \Longrightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{L}$

State (=**probability measure**) $s\colon \mathcal{L} \to [0,1]$:

- $s(U) = 1$
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finitely, resp. **countably** additive for $\mathbb{I}$ finite, resp. countable

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State space (of **finitely** additive states): $\mathcal{S}(\mathcal{L}) \subseteq [0,1]^{\mathcal{L}}$ compact, convex, Choquet simplex

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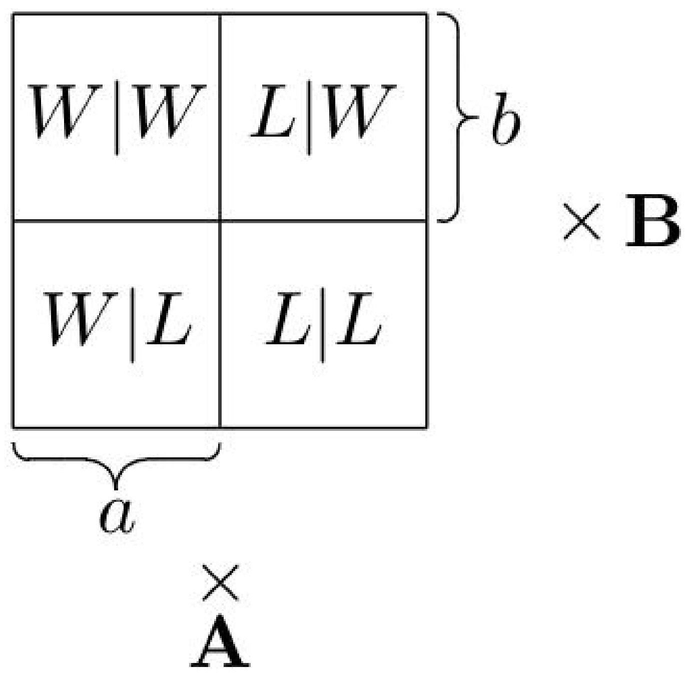
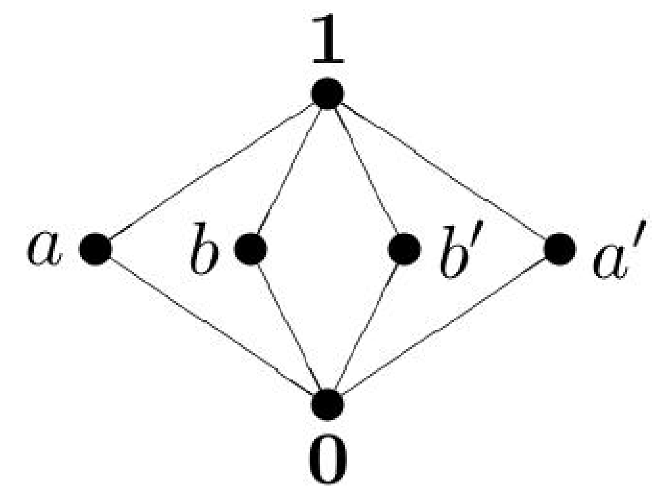
Example 1 (transparent barriers)

W/L = Wins/Loses with|without player JJ





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$\mathcal{L} = A \cup B = \{\emptyset, a, a', b, b', U\}$

$\mathcal{L}$ is a (nondistributive modular) lattice called $MO2$

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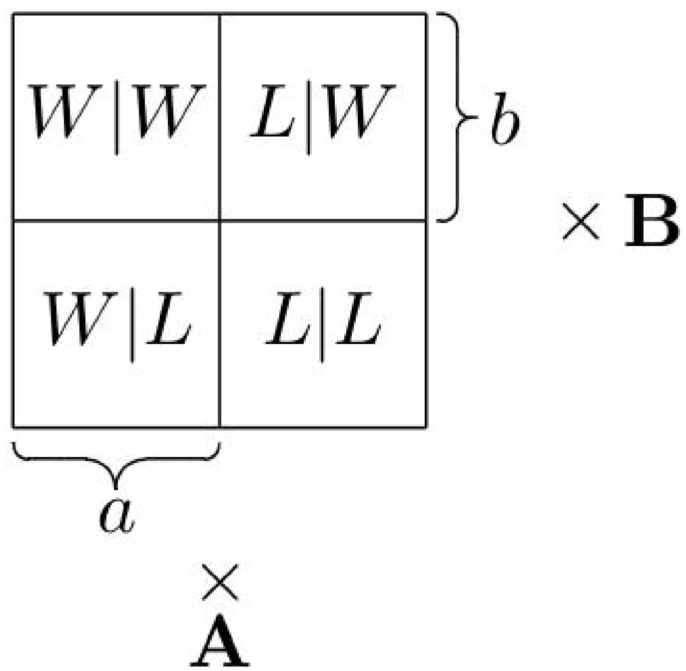

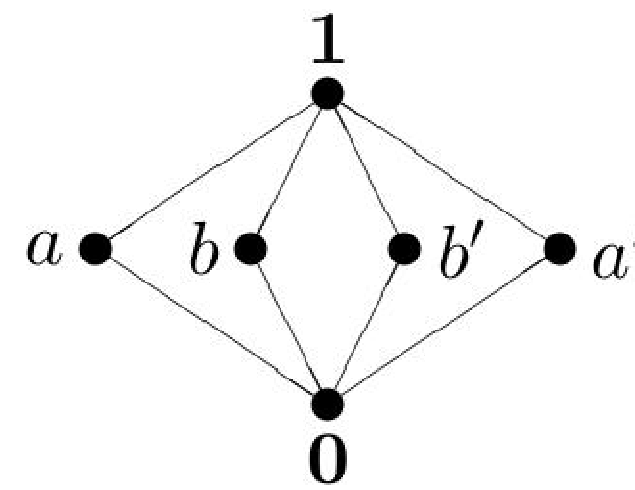

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Example 2 (transparent barriers)

Example 1 with one more result, $c =$ match cancelled

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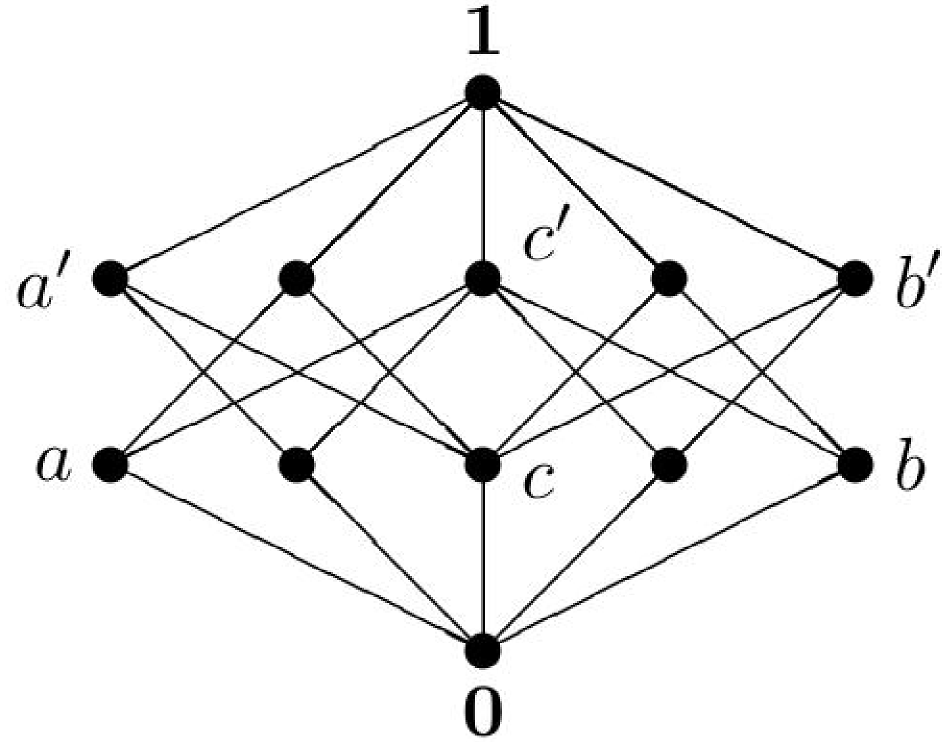

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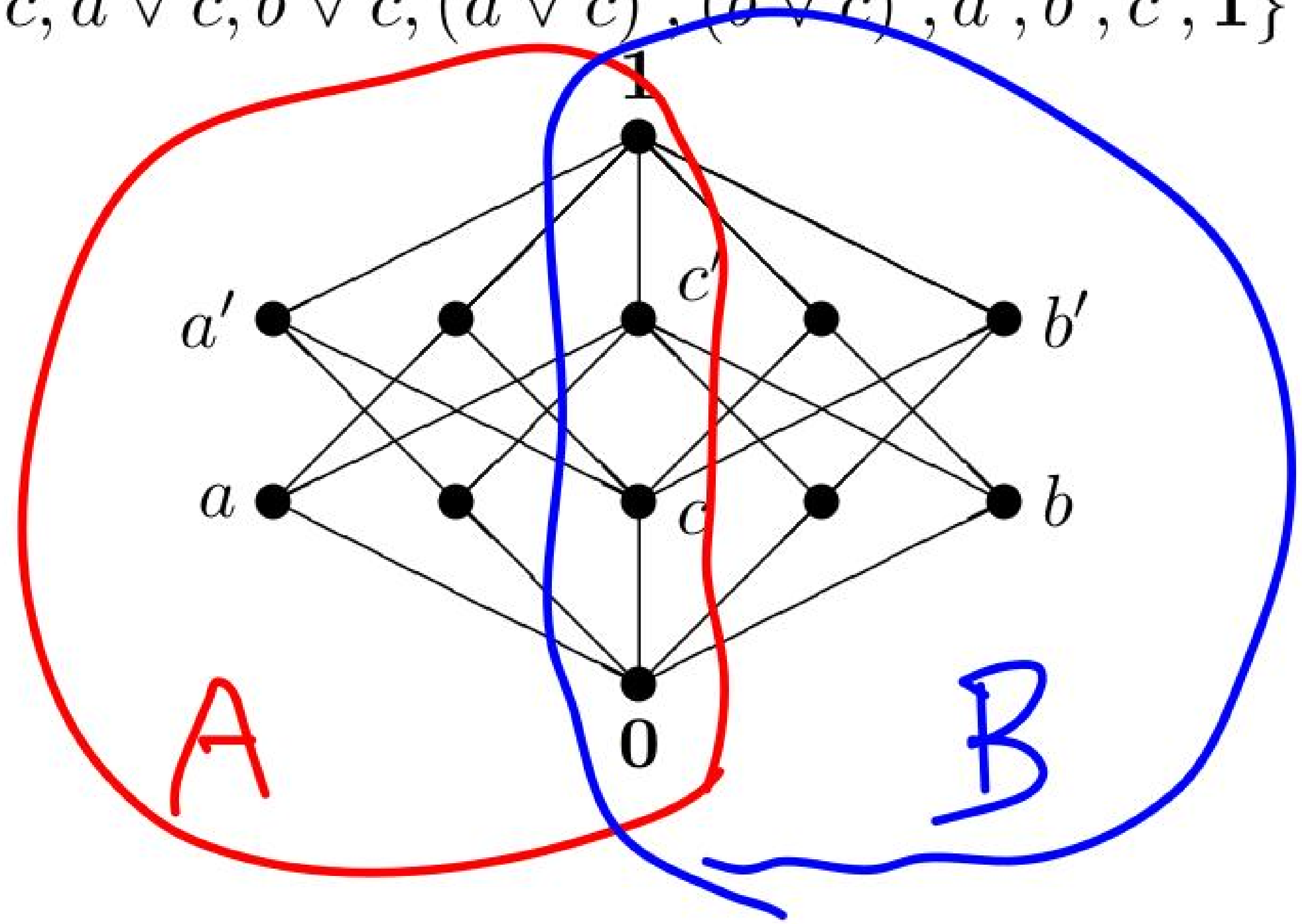
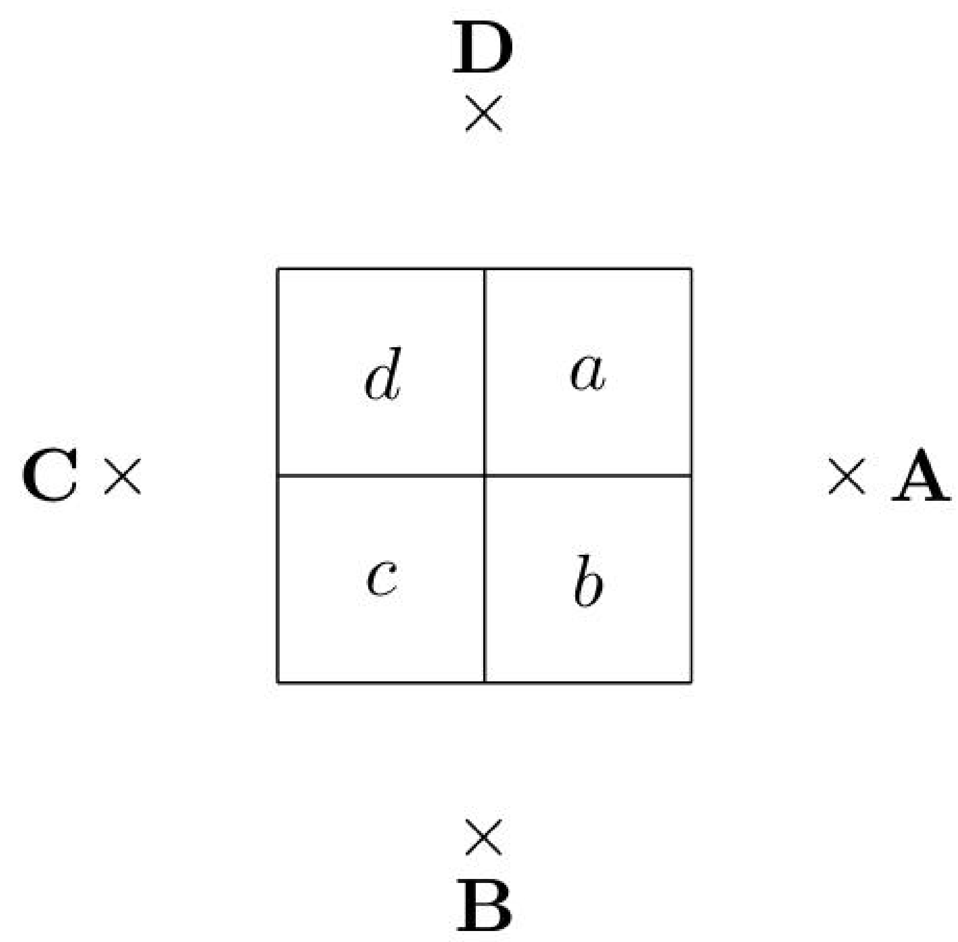
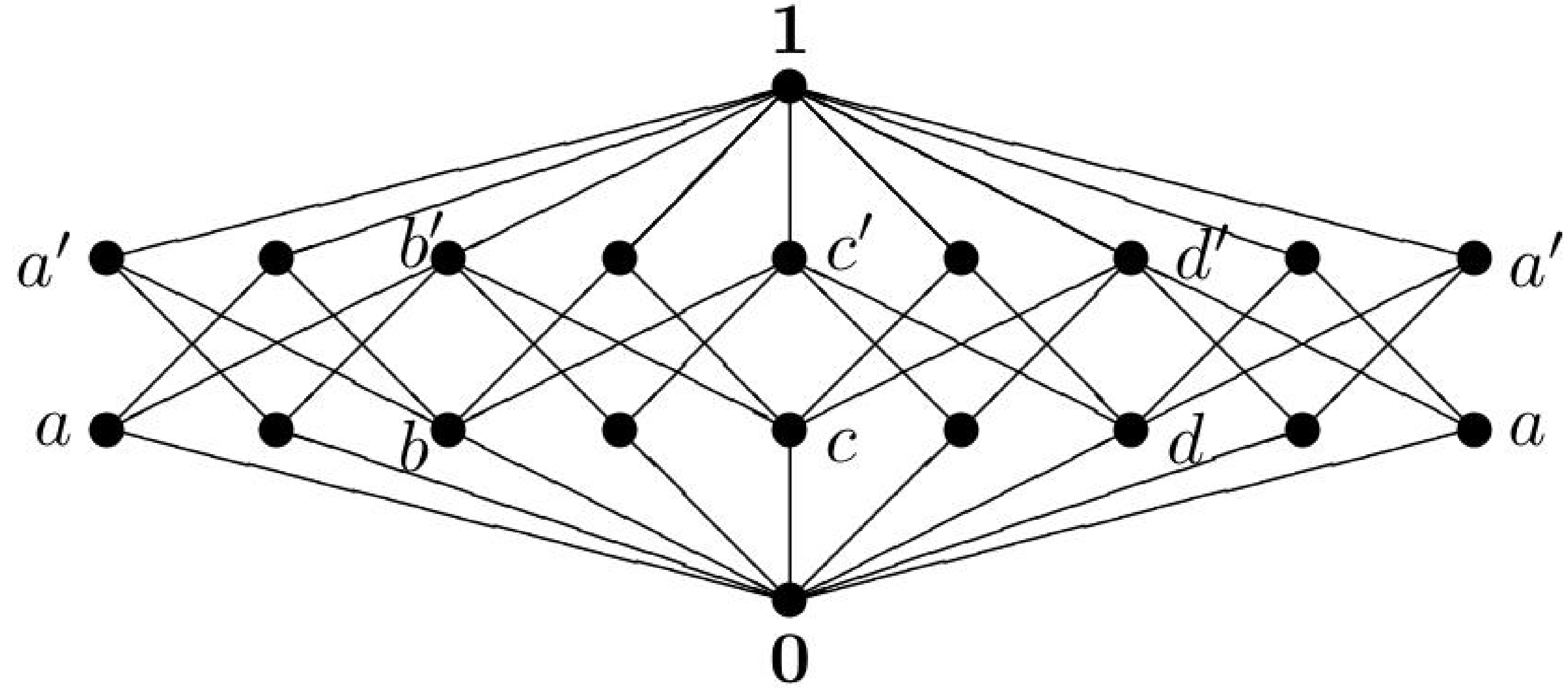
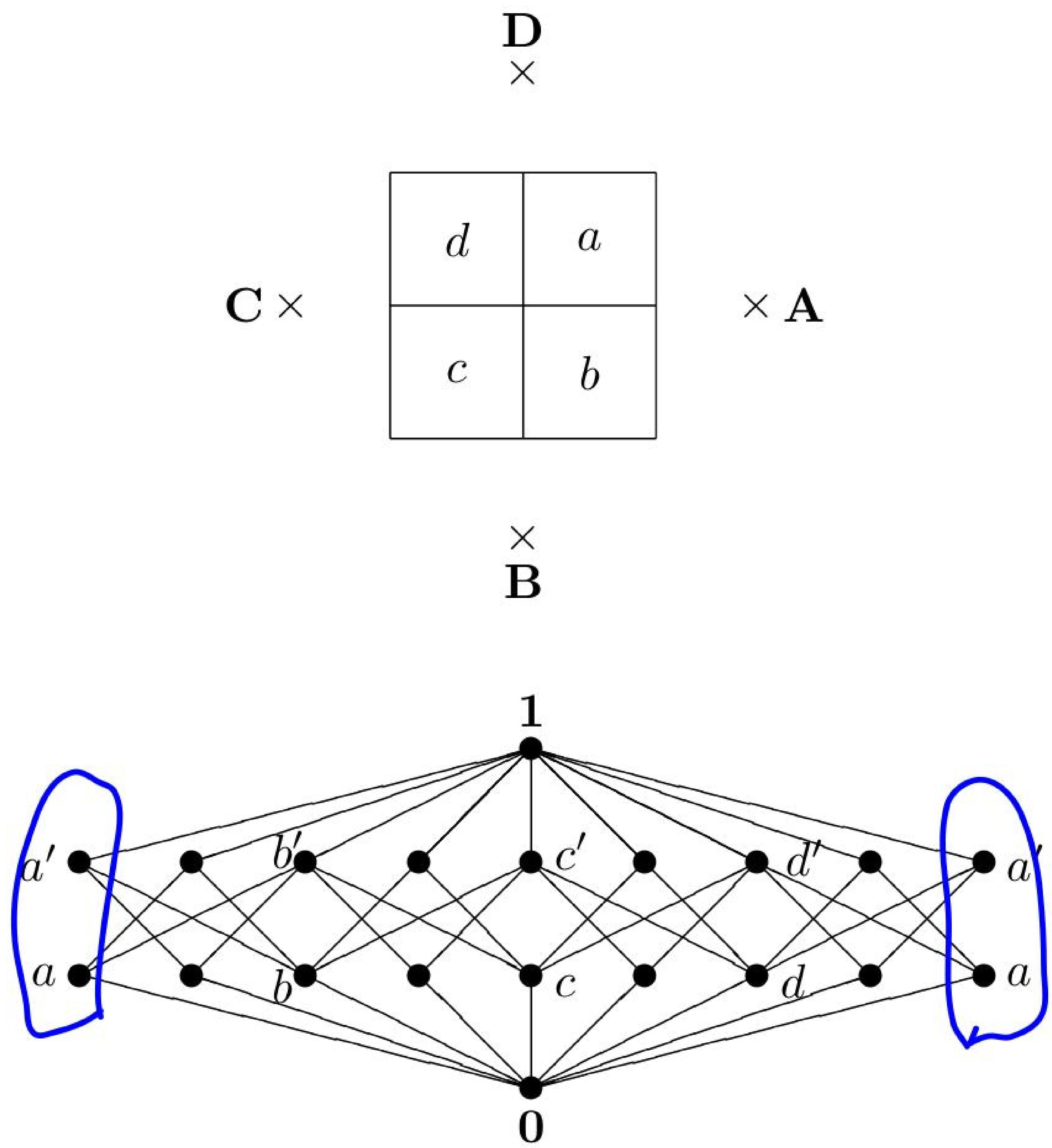


Example 3 (non-transparent barriers)






D
C
A
B
d
a
c
b
1
0
a′
a
b′
b
c′
c
d′
d

Definition: An **orthomodular poset** is a poset with bounds $\mathbf{0}$, $\mathbf{1}$ equipped with a unary operation $' : \mathcal{L} \to \mathcal{L}$ (**orthocomplementation**) such that, for all $a, b \in \mathcal{L}$,

- $a'' = a$
- $a \leq b \Longrightarrow b' \leq a'$
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Compatibility: $a \leftrightarrow b \iff \exists$ Boolean subalgebra $\mathcal{M}$: $a, b \in \mathcal{M}$

Atom: $a \in \mathcal{L} \setminus \{\mathbf{0}\}$ such that there is no b satisfying $\mathbf{0} < b < a$
$\mathcal{A}(\mathcal{L}) :=$ the set of all atoms of $\mathcal{L}$

Proposition: The space of finitely additive states is convex and compact (in the product topology).

Theorem: [Shultz 74, MN 92] **Every** compact convex subset of a locally convex topological linear space is affinely homeomorphic to the space of finitely additive (as well as all) states of some OML.

Characterization of state spaces — finitely additive case

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This is a mathematical peculiarity, for physical applications we need "enough" states:

Definition: An OMP L is **rich** iff $\forall a, b \in L, a \not\leq b$ $\exists$state $s : s(a) = 1 = s(b)$.

Theorem: All rich OMLs form a variety.

$s(a) + s(b) - s(a \wedge b) \leq 1$

$0 \geq s(a \wedge b) + s(b \wedge c) + s(c \wedge d) - s(a \wedge d) - s(b) - s(c)$

$s(a) + s(b) + s(c) - s(a \wedge b) - s(a \wedge c) - s(b \wedge c) \leq 1$

$s(a \wedge b) + s(b \wedge c) + s(c \wedge d) - s(a \wedge d) - s(b) - s(c) \geq -1$

The first is equivalent to the **valuation property**:

$s(a \wedge b) + s(a \vee b) = s(a) + s(b)$

If a rich OML is not a Boolean algebra, all Bell inequalities are violated.

Crucial example of a quantum structure: Hilbert lattice

H ... a separable Hilbert space (real or complex)
$L(H)$ ... the set of all closed subspaces of H (equivalently, all projectors of H)

$$\begin{aligned}
\mathbf{0} &= \{0\}, \\
\mathbf{1} &= H, \\
A \le B &\iff A \subseteq B, \\
A \wedge B &= A \cap B, \\
A' &= \{x \in H \mid \forall y \in A : y \perp x\}, \\
A \vee B &= \mathit{Lin}(A \cup B),
\end{aligned}$$

where *Lin* denotes the closed linear hull

For projectors, compatibility is equivalent to commutation, $PQ = QP$, therefore the term "*noncommutative measure theory*"

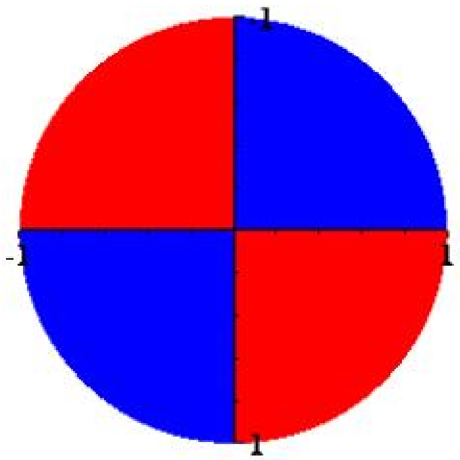
Problem [Birkhoff, von Neumann, Mackey]: Find (well-motivated, verifiable) properties which characterize Hilbert lattices among OMLs.

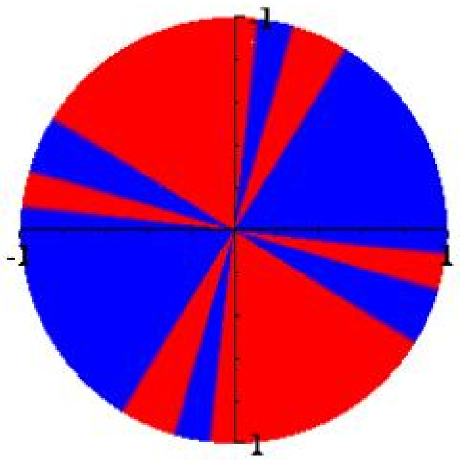
Partial results: [Pulmannová], [Bunce, J.D.M. Wright] ...

States on Hilbert lattices in $\mathbb{R}^2$

The only restriction of states for $\dim P = 1$: $\quad s(P') = 1 - s(P)$

Many two-valued states = colourings of non-zero vectors by two colors (blue, red) such that each orthogonal basis contains exactly one red vector





1. For $q \in H$, $||q|| = 1$, define a **vector state**

$$s_q(\mathit{Lin}(\{y_1, \dots, y_n\})) = \sum_{i=1}^{n} (q \cdot y_i)^2 = \sum_{i=1}^{n} \cos^2 \sphericalangle(q, y_i)$$

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Gleason's Theorem [Gleason 57]: For $\dim H \geq 3$, all states are mixtures of vector states.

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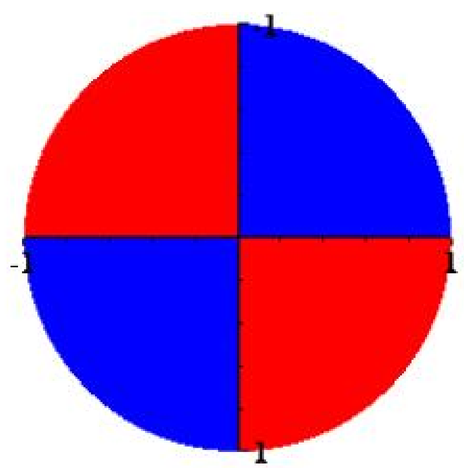
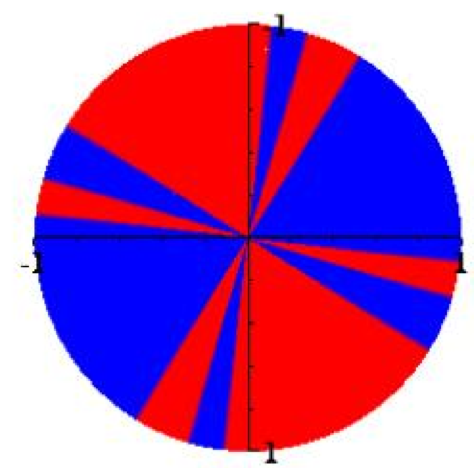
The only restriction of states for $\dim P = 1$: $\qquad s(P') = 1 - s(P)$

Many two-valued states = colourings of non-zero vectors by two colors (blue, red) such that each orthogonal basis contains exactly one red vector





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Crucial case: $H = \mathbb{R}^3$ (simplified proof by [Cooke, Keane, Moran 85]).

Corollary 1: The restriction of a state to 1D subspaces is continuous (proved by [von Neumann 1932] even for $\mathbb{R}^2$, error found by [Hermann 1935],

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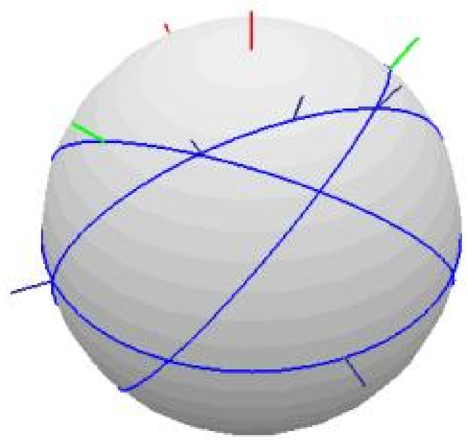
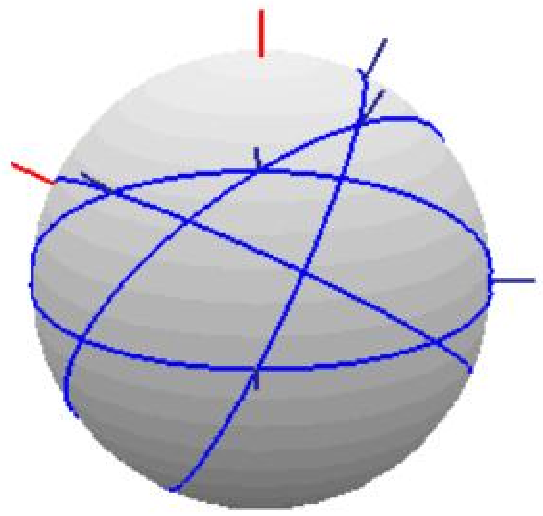
BGL

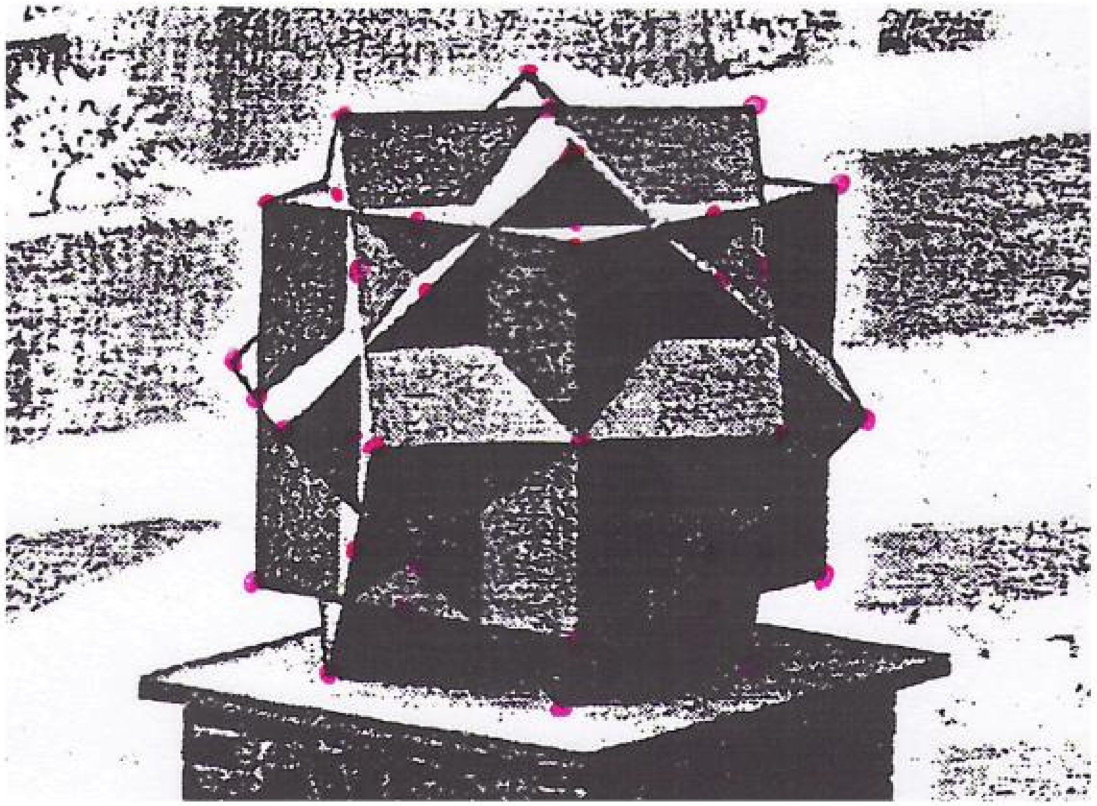


MGL



It is possible to find a finite set of vectors whose orthogonality relations exclude the possibility of a two-valued state.

The smallest example known uses 31 vectors, the following uses 33 vectors:



Theorem: [Cabello] There is no two-valued state on $\mathcal{L}(\mathbb{R}^4)$.

Take 36 vectors in $\mathbb{R}^4$ ($\bar{1}$ denotes -1):

1000	1000	0100	1111	1111	$111\bar{1}$	$11\bar{1}\bar{1}$	$111\bar{1}$	$11\bar{1}1$
0100	0010	0010	$11\bar{1}\bar{1}$	$1\bar{1}1\bar{1}$	$11\bar{1}1$	$1\bar{1}1\bar{1}$	$1\bar{1}11$	$1\bar{1}11$
0011	0101	1001	$1\bar{1}00$	$10\bar{1}0$	$1\bar{1}00$	1001	$10\bar{1}0$	$100\bar{1}$
$001\bar{1}$	$010\bar{1}$	$100\bar{1}$	$001\bar{1}$	$010\bar{1}$	0011	0110	0101	0110

Each of the 9 column represents an orthogonal basis of $\mathbb{R}^4$ and each vector **occurs twice**. The number of vectors of unit state in this table must be both even and odd (9)—a contradiction.

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Theorem: ([Pták, MN 04], extended by [Harding, Jager, Smith])
If $n \geq 4$, then there is no non-trivial $\mathbb{Z}_2$-valued **state** on $\mathcal{L}(\mathbb{R}^n)$.

Another proof in [Peres 1996].

Constructions proving the non-existence of two-valued states in $\mathbb{R}^4$

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Answer: NO. [Voráček & MN, 2021].

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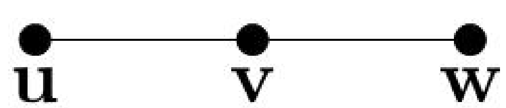
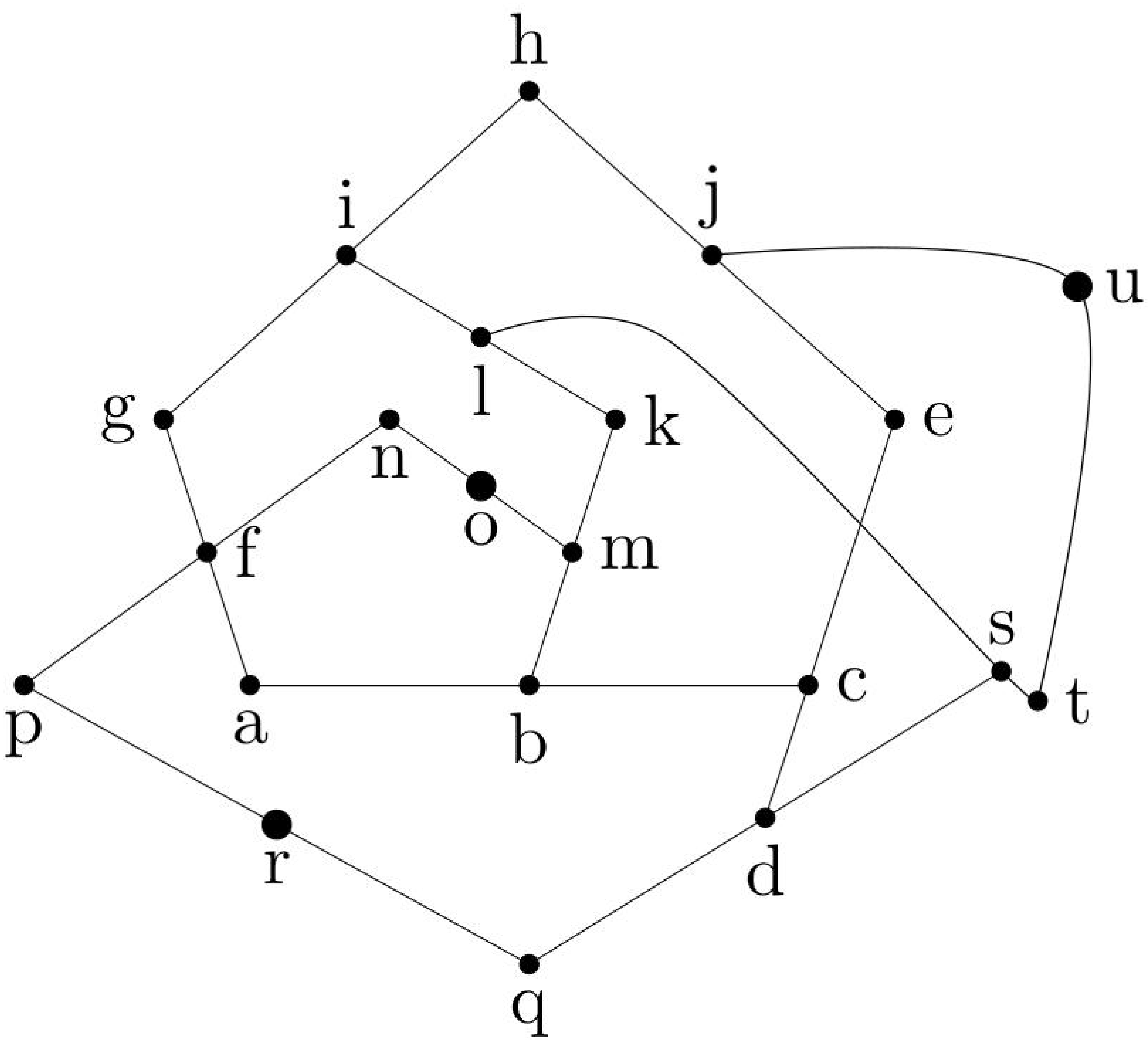
Notation of an orthogonal basis:



Each such edge must contain an odd number of "red" vertices (vectors evaluated by 1).

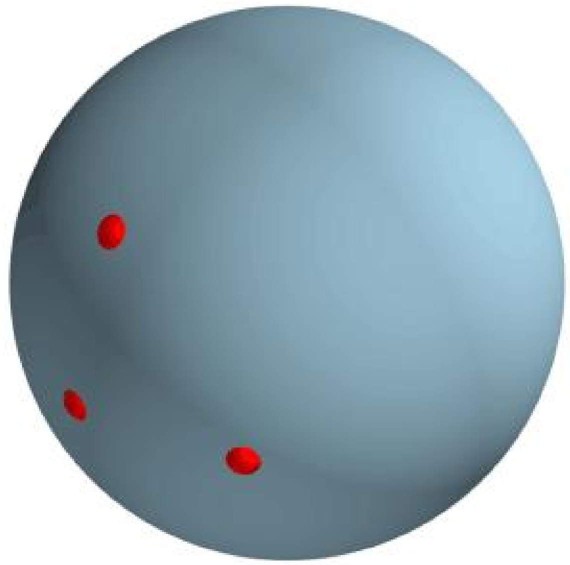


There are vectors in $\mathbb{R}^3$ satisfying these orthogonality relations (Greechie diagram). There are 13 edges. Except for $\mathbf{r}, \mathbf{u}, \mathbf{o}$, each vector is contained in 2 edges. Vectors $\mathbf{r}, \mathbf{u}, \mathbf{o}$ are not orthogonal, but satisfy the same equation:

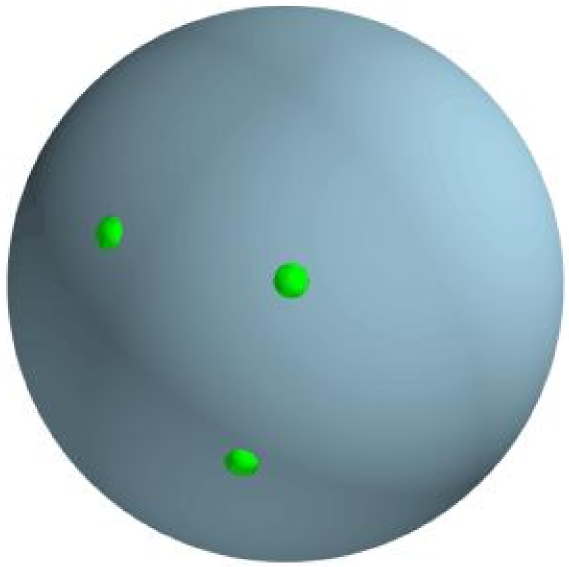
$$\overbrace{s(\mathbf{a}) \oplus s(\mathbf{b}) \oplus s(\mathbf{c})}^{=1} \oplus \overbrace{s(\mathbf{a}) \oplus s(\mathbf{f}) \oplus s(\mathbf{g})}^{=1} \oplus \cdots = 13 = 1 \pmod 2$$

$$= \quad s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o})$$

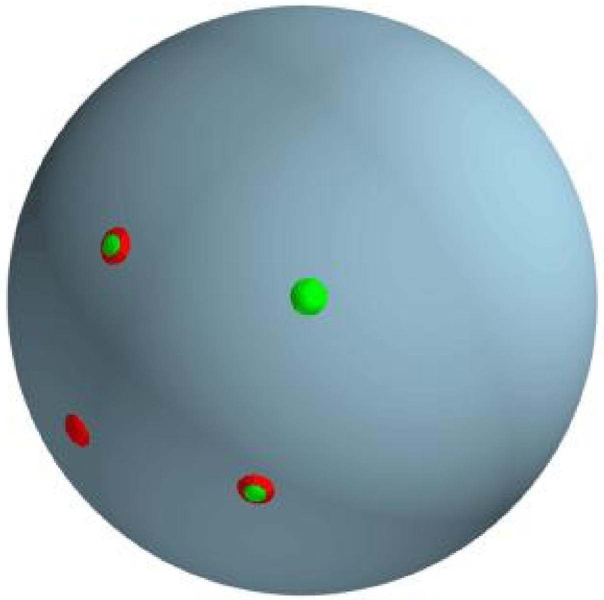
New proof of non-existence of $\mathbb{Z}_2$-valued states on $L(\mathbb{R}^3)$



$$s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o}) = 1$$



$$s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o}') = 1$$



$$\overbrace{s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o})}^{1} \oplus \overbrace{s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o}')}^{1} = 0 = s(\mathbf{o}) \oplus s(\mathbf{o}')$$

$$s(\mathbf{o}) = s(\mathbf{o}')$$